A common theme in mathematics is that linear things are easier to study than non-linear ones. Thus, when faced with a problem, one hopes to be able to convert it into a linear one, without losing too much information in the process. Geometric quantisation is one way of converting a group action on a suitable manifold into a representation of that group on a vector space; I study the characters of these representations.

Let G be a compact connected Lie group. The Weyl Character Formula gives the character of an irreducible representation of G in terms of its highest weight. By considering a partition function which counts the number of ways an integral weight can be expressed as an integer combination of positive roots, Kostant ([6]) derived a formula for the multiplicity with which a given torus weight occurs in the irreducible G-representation of highest weight λ.

Suppose $K \subset G$ is a compact subgroup, and that K and G have a common maximal torus T. In [3], Gross, Kostant, Ramond and Sternberg gave a formula for the character of an irreducible G-representation as a quotient of characters of K-representations.

In my paper [2] I generalised their result, obtaining a formula for the character χ of the quantisation of an arbitrary compact connected symplectic Hamiltonian G-manifold, as a quotient of K-characters. Using a modified version of Kostant's partition function I derived from my character formula a formula for the multiplicity with which irreducible characters of K appear in χ; this multiplicity formula generalises the one appearing in Guillemin and Prato's paper [5]. This work is described in more technical detail below.

Let (M, ω) be a compact connected symplectic manifold. Suppose G acts on M in a Hamiltonian manner, with moment map $\mu : M \to \mathfrak{g}^*$, and that action of T on M obtained by restriction has isolated fixed points. A symplectic manifold is said to be *prequantisable* when the equivariant cohomology class $\frac{1}{2\pi}[\omega - \mu]$ is integral; in this case there is a line bundle $L \to M$, known as a prequantisation line bundle, with first equivariant curvature class $\omega - \mu$. Pick a G-invariant connection ∇ on L whose equivariant curvature form is $\omega - \mu$, and a G-equivariant almost complex structure J on M which is compatible with ω. Then the quantisation $Q(M, \omega, \nabla, L; J)$ is defined as the equivariant index of the operator $\overline{\partial} + \overline{\partial}^t : \Omega^{0,even}(M; L) \to \Omega^{0,odd}(M; L)$. That is, the (virtual) G-representation

$$Q(M, \omega, \nabla, L; J) = \ker(\overline{\partial} + \overline{\partial}^t) \ominus \operatorname{coker}(\overline{\partial} + \overline{\partial}^t).$$

See [4] for further details. My research focuses on the characters $\chi(Q)$ of these representations.

The Weyl Character Formula and Kostant's multiplicity formula arise as special cases as follows. Suppose G is semisimple, let λ be a dominant integral weight for G, and take M to be the coadjoint orbit $G \cdot \lambda \cong G/T$, with G acting by left multiplication. Take as the prequantisation line bundle $L_\lambda = (G \times \mathbb{C}_{(\lambda)})/T$, where T acts on G by (right) multiplication, on $\mathbb{C}_{(\lambda)}$ with weight λ, and on the product by the diagonal action. In this case the quantisation is a G-representation on the space of holomorphic sections of L_λ (see [4], chapter 6). By the Borel-Weil theorem, this quantisation is the irreducible G-representation of highest weight λ; furthermore, all irreducible G-representations arise in this way. The character of this representation is given by the Weyl Character Formula (as described by Bott in [1]), and the multiplicities of the weights appearing in this character are given by the Kostant Multiplicity Formula.

Precise statement of results

Choose a set $\Phi^+(G)$ of positive roots of G, and let $\mathcal{W}_G \subset \mathfrak{t}^*$ be the positive Weyl chamber for G. Let $\Lambda \subset \mathfrak{t}^*$ denote the weight lattice. Let $P = \{p_1, \ldots, p_n\}$ be the set of fixed points of the torus action on M. Let P^+ be the set of fixed points which map into the closed positive Weyl chamber $\overline{\mathcal{W}_K}$ under the moment map.

Let S be the maximal torus of the centraliser of K in G. Given a fixed point $p_i \in M$, let F be the connected component of the fixed set M^S containing p_i. Let B_i be the set of weights of the torus action on the normal bundle $N_{p_i}(K \cdot p_i)$ to the K-orbit of p_i at p_i. We will polarise the weights as follows. Choose a $\xi \in \mathfrak{t}$ such that $\langle \alpha, \xi \rangle = 0$ for all $\alpha \in W(K)$, and $\langle \lambda, \xi \rangle \neq 0$ for all other weights λ of the torus action on tangent spaces to fixed points. So $W(K)$ preserves ξ. Given a weight β, define its *polarisation* by

$$\beta^+ = \begin{cases} \beta, & \beta(\xi) \geq 0 \\ -\beta, & \beta(\xi) < 0 \end{cases},$$

and for fixed i, let $s_i = |\{\beta \in B_i \mid \beta(\xi) < 0\}|$.

For the fixed point p_i, write $\beta_i = \frac{1}{2} \sum_{\beta \in B_i} \beta$. Let $\mathcal{B}_i$ be the set $\{w \cdot \beta \mid w \in W(K), \beta \in B_i\}$, and let $\mathcal{B}_i^+$ be the set of polarised weights $\{\beta^+ | \beta \in \mathcal{B}_i\}$. Let $\overline{\beta_i} = \sum_{\beta \in B_i : \beta = -\beta^+} \beta$. Let C_i be the set of weights $\alpha \in \mathcal{B}_i^+$ that are not equal to β^+ for any $\beta \in B_i$; that is, polarisations of all weights in the orbit except for those at the point p_i. Let $m_i(\eta)$ be the multiplicity of e^η in $\prod_{\gamma \in C_i}(1 - e^{-\gamma})$.

In [2], I used the Equivariant Index Theorem and the Atiyah-Bott-Berligne-Verne Localisation Formula to prove the following generalisation of the Gross-Kostant-Ramond-Sternberg result to our setting.

Theorem 0.1 (Gamse, [2])**.** *Let G be a compact Lie group with maximal torus T. Let $(M, \omega, L, \nabla; J)$ be a Hamiltonian G-space such that the T-action has isolated fixed points. Let $K \subset G$ be a closed connected subgroup with maximal torus T. Write V_λ^K for the irreducible K-representation of highest weight λ. Then the character χ of the quantisation $Q(M, \omega, L, \nabla; J)$ is given by*

$$\chi(Q(M, \omega, \nabla, L; J)) = \sum_{F \in \pi_0(M^S)} \sum_{p_i \in F \cap P^+} \frac{\sum_{\eta \in \Lambda} (-1)^{s_i} m_i(\eta) \chi(V^K_{\mu(p_i) + \overline{\beta_i} + \eta})}{\prod_{\gamma \in \mathcal{B}_i^+}(1 - e^{-\gamma})}.$$

Instead of torus weights, one can also ask with what multiplicity irreducible K-characters appear in the G-representation $Q(M, \omega, \nabla, L; J)$. For this we introduce the function $P_i(\zeta) : \mathfrak{t}^* \to \mathbb{N}$ which counts the number of ways to write ζ as a sum $\sum_{\beta \in B_i} c_\beta \beta^+$, where the c_j are non-negative integers. Let ρ_K denote half the sum of the positive roots of K. Let Z_i denote the set of pairs (p, w) where $p = w \cdot p_i$ for $w \in W(K)$. Note that $|Z_i| = |W(K)|$. We let Z be the union over all $p_i \in P^+$ of the Z_i; that is, let Z be the set of pairs (p, w) where p is a fixed point of the T action on M and $w \in W(K)$ is such that

$w^{-1}(\mu(p))$ is in the closed positive Weyl chamber $\overline{\mathcal{W}_K}$. So each regular point appears once; the non-regular points appear once for each Weyl chamber in whose boundaries they lie. Thus the following formula, which I derive from Theorem 0.1 in [2], is a generalisation of the Guillemin-Prato formula (which requires that all fixed points be regular).

Theorem 0.2 (Gamse, [2])**.** *Let $K \subset G$ be compact Lie groups of equal rank with a common maximal torus T. Let $(M, \omega, L, \nabla; J)$ be a compact Hamiltonian G-space, and let $Q(M, \omega, \nabla, L; J)$ be its quantisation. Let λ be a dominant weight for K. Then the multiplicity of the irreducible K-representation of highest weight λ in the G-representation $Q(M, \omega, \nabla, L; J)$ is given by*

$$\#(\lambda, Q(M, \omega, \nabla, L; J)) = \sum_{(p_i, w) \in Z} (-1)^{s_i} \epsilon(w) P_i(\mu(p_i) + w \cdot \rho_K + \beta_i - \lambda - \rho_K - \beta_i^+).$$

Theorem 0.2 is an example of the power of combinatorial techniques in the study of symplectic geometry.

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