Practice questions for the final exam

MAT392, Winter 2019

March 29, 2019

The questions on the exam will be based on the in-class activities and discussions, the homework assignments, the assigned readings, and questions from this document. Assigned reading included:

- Zeno's Paradox, from *Math Academy Online*
- Limits: Tim Gowers, *Princeton Companion to Mathematics pp30-32*
- Equivalence Relations: Eugenia Cheng, *How to bake* π
- Sets, functions, and the continuum hypothesis: Aigner & Ziegler, *Proofs from the Book*
- Infinite and Infinitesimal Numbers: from Conway & Guy, *The Book of Numbers*
- An easy proof of the fundamental Theorem of Algebra: Charles Fefferman, *The American Mathematical Monthly*
- extract from Benjamin Fine & Gerhard Rosenberger, *The Fundamental Theorem of Algebra*
- Some writing tips: from Vivaldi, *Mathematical Writing for undergraduate students*
- Cliques and Anticliques: from Peter Hilton, Derek Holton, Jean Pedersen, *Mathematical Vistas from a Room with Many Windows*
- Constructibility and Number Fields:from Courant and Robbins, *Mathematics: People, Problems, Results*
- The isoperimetric inequality: Polya

"Unseen" Reading

You will be given a short text that you have not seen before. You may be asked to write a short report about the text. You may be asked to answer some questions about the text.

Sample Questions

1. Write a two-paragraph mini-essay about...
    - Zeno's paradox
    - Infinities
    - Doubling the Cube
    - Straightedge and Compass Constructions
    - The isoperimetric inequality
    - Constructible numbers
    - The Cantor-Schröder-Bernstein Theorem
    - Countability
    - Ordinal and Cardinal numbers
    - The fundamental theorem of algebra
    - Ramsey theory
2. Give a definition, an example, and a non-example for each of the following:
    - Convergent sequence (of real numbers)
    - Divergent series (of real numbers)
    - Absolutely convergent series
    - Conditionally convergent series
    - Relation on a set X
    - Equivalence relation
    - Partial order
    - Cardinality of a set
    - Similarity of well-ordered sets
    - Uncountable set
    - Algebraic Number
    - Field
    - Quadratic Field Extension
    - Constructible Number
    - Complete graph
3. Give precise statements of
    a) The Cantor-Schröder-Bernstein Theorem
    b) Ramsey's Theorem

c) The infinite Ramsey theorem

d) The Fundamental Theorem of Algebra

e) The Isoperimetric Inequality

4. Complete each sentence by an appropriate single word or phrase.

a) If $\{a_n\}$ is a sequence and $\forall N \in \mathbb{N}\ \exists n > N$ such that $a_n > N$, then a_n ...

b) If $\{a_n\}$ is a sequence and $\exists N \in \mathbb{N}$ such that $\forall n > N \quad a_n > N$, then a_n is...

c) If $\{a_n\}$ is a sequence and $\forall \epsilon > 0\ \exists N \in \mathbb{N}$ such that $\forall n > N$, $|a_n| < \epsilon$, then a_n ...

d) If $\{a_n\}$ is a sequence and $\exists N$ such that $\forall \epsilon > 0\ \forall n > N \quad |a_n| < \epsilon$ then a_n ...

e) If $n \in \mathbb{Z}$ and, $\forall m \in \mathbb{Z}$ with $m > 0$, $\ -m < n < m$, then n is...

f) If P is a set and S is a set such that $x \in P \Leftrightarrow x \subseteq S$, then P is the...

g) If f is a complex polynomial and $\forall x\ f(x) \neq 0$ then f is ...

5. Below we list properties that a function $f\colon \mathbb{R} \to \mathbb{R}$ might have. Determine which of these properties implies which other property. For each two of these properties, give an example of a function that has one of the two properties but not the other. (You need to give $\binom{4}{2}$ examples. You may use the same function more than once.)

a) For every a and b in $\mathbb{R}$, if $a < b$ then $f(a) < f(b)$.

b) For every a and b in $\mathbb{R}$, if $f(a) < f(b)$ then $a < b$.

c) For every a and b in $\mathbb{R}$, if $a \neq b$ then $f(a) \neq f(b)$.

d) For every a and b in $\mathbb{R}$, if $f(a) \neq f(b)$ then $a \neq b$.

6. True or false? Explain. Give a proof or a counterexample.

a) Every algebraic number is irrational.

b) Every rational number is algebraic.

c) Every algebraic number is rational.

d) Every constructible number is rational.

e) Every algebraic number is constructible.

f) Every real number is algebraic.

g) If $P(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ is a non-constant polynomial with coefficients $a_0, \ldots, a_n$ in a field F then $\exists x \in F$ such that $P(x) = 0$.

h) Every real number is the zero of some polynomial.

Sequences

7. a) Define "the sequence $\{a_n\}$ converges."
   Show, from the definition, that the sequence $a_n = \frac{\sin n}{n}$ converges.

   b) Define "the series $\sum_{n=1}^{\infty} a_n$ converges".
   Show, from the definition, that the series $1 + \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \dots$ converges.

   c) Define "the sequence $\{a_n\}$ does not converge" without using the word "not".
   Show, from the definition, that the sequence $a_n = (-1)^n + \frac{1}{n}$ does not converge.

   d) Define "the series $\sum_{n=1}^{\infty} a_n$ does not converge" without using the word "not".
   Show, from the definition, that $1 + 10 + 10^2 + 10^3 + \dots$ does not converge.

   e) Define "the series $\sum_{n=1}^{\infty} a_n$ is conditionally convergent".

8. Below we list examples of properties that a sequence $\{b_n\}_{n=1}^{\infty}$ of real numbers might have, and examples of sequences $\{b_n\}_{n=1}^{\infty}$ of real numbers.

   What do the properties mean? Which of the properties implies which other? Which of the sequences has which of the properties?

   Property 1: For all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, $|b_n| < \epsilon$.
   Property 2: There exists $N \in \mathbb{N}$ such that, for all $\epsilon > 0$, for all $n \geq N$, $|b_n| < \epsilon$.
   Property 3: There exists $\epsilon > 0$ such that, for all $N \in \mathbb{N}$, for all $n \geq N$, $|b_n| < \epsilon$.
   Property 4: For all $N \in \mathbb{N}$ there exists $\epsilon > 0$ such that, for all $n \geq N$, $|b_n| < \epsilon$.

   Sequence 1: $b_n = \begin{cases} \frac{1}{n} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$.
   Sequence 2: $b_n =$ the nth digit of π after its decimal mark.
   Sequence 3: $b_n = \frac{1000}{n}$.
   Sequence 4: $b_n = \lfloor \frac{1000}{n} \rfloor$.
   Sequence 5: $b_n = \begin{cases} n & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$.

9. Here are four properties that a sequence $\{a_n\}$ of real numbers might have.

   a) The sequence is bounded.

   b) Eventually, (i.e., for large enough n,) all the elements of the sequence are zero.

   c) The sequence contains infinitely many zeros.

   d) The sequence has a bounded subsequence.

Express each of these properties by combining some of the following phrases in some order:
"for all $b > 0$", "there exists $b > 0$ such that",
"for all $N \in \mathbb{N}$", "there exists $N \in \mathbb{N}$ such that",
"for all $n \geq N$", "there exists $n \geq N$ such that",
"$|a_n| < b$" .

Relations, equivalence classes, and well-defined operations

10. Let $\sim$ be a binary relation that satisfies the following two properties.

    (C.N.1) for all x, y, z, if $x \sim z$ and $y \sim z$ then $x \sim y$.

    (C.N.4) for all x, we have $x \sim x$.

    Prove that, for all a, b, if $a \sim b$ then $b \sim a$.
    Prove that, for all a, b, c, if $a \sim b$ and $b \sim c$ then $a \sim c$.
    (Thus, $\sim$ is an equivalence relation.)

11. Prove that $x - y \in 2\pi\mathbb{Z}$ defines an equivalence relation on $\mathbb{R}$. (The equivalence classes are called "real numbers modulo 2π".) Prove that addition is well defined on real numbers modulo 2π. Prove that multiplication is not well defined on real numbers modulo 2π.

12. Describe the rational numbers as the equivalence classes for an equivalence relation on certain pairs of integers.

13. For sets A, B, define $A \triangle B = (A \setminus B) \cup (B \setminus A)$. Is this binary operation commutative? Is it associative? Explain.

14. Consider subsets of $\mathbb{N}$. Recall that the symmetric difference of such subsets A and B is $A \triangle B := (A \setminus B) \cup (B \setminus A)$. Is this operation commutative? Is it associative? Show that it has a neutral element. Does every element have an inverse?

15. Consider subsets of $\mathbb{N}$. For each of the following four relations on such subsets, determine which of the following properties it has: reflexive, symmetric, transitive, anti-symmetric.

    Relation 1: There exists a map from A to B that is one-to-one.
    Relation 2: There exists a map from A to B that is onto.
    Relation 3: The symmetric difference $A \triangle B$ is finite.
    Relation 4: A contains the complement of B.

16. Which of the following operations is well defined? Explain.

    - Operation on cardinalities: $|A| + |B| = |A \cup B|$.
      Here, A and B are arbitrary subsets of $\mathbb{R}$ and $|A|$ and $|B|$ are their cardinalities.

- Operation on cardinalities: $|A| \cdot |B| = |A \times B|$.
  Here, A and B are any sets and $|A|$ and $|B|$ are their cardinalities.
- Operation on positive rational numbers: $\frac{a}{b} \star \frac{c}{d} = \frac{a+c}{b+d}$.
- Operation on positive rational numbers: $\frac{a}{b} \star \frac{c}{d} = \frac{a \cdot d}{b \cdot c}$.
- Operation on positive rational numbers: $\frac{a}{b} \star \frac{c}{d} = \frac{bd}{ad+bc}$.

17. a) Define a relation on $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$: $(a, b) \sim (b, d)$ if $ad = bc$. Show that it is an equivalence relation. Denote by $[(a, b)]$ the equivalence class of (a, b). Is the operation $[(a, b)] \star [(c, d)] := [(ac + bd, bd)]$ well defined? Why or why not?

    b) Define a relation on $\mathbb{Z} \times \mathbb{Z} \setminus \{(0, 0)\}$: $(a, b) \sim (b, d)$ if $ad = bc$. Show that it is an equivalence relation. Denote by $[(a, b)]$ the equivalence class of (a, b). Is the operation $[(a, b)] \star [(c, d)] := [(ac + bd, bd)]$ well defined? Why or why not?

Order relations

18. On the set $\mathbb{N} \times \mathbb{N}$, define: $(x_1, y_1) \leq (x_2, y_2)$ if either
    - $x_1 < x_2$, or
    - $x_1 = x_2$ and $y_1 \leq y_2$.

    Prove that this relation is a partial ordering.

19. Let $\leq$ be the usual order on $\mathbb{N}$. Define a nonstandard order $\leq'$ on $\mathbb{N}$ as follows. For $n, m \in \mathbb{N}$, declare $n \leq' m$ if and if n is odd and m is even, or n, m are both odd and $n \leq m$, or n, m are both even and $n \leq m$.

    Prove that this relation is a total order. Prove that is a well ordering.

20. Let $\leq$ be a total ordering on a set. Show that, if there exists a minimal element, then this element is unique.

21. For each of the following subsets of $\mathbb{R}$, determine if its standard order, induced from $\mathbb{R}$, is a well ordering. Justify.
    (a) $[0, \infty)$,
    (b) $\{n - \frac{1}{m}\}_{n,m \in \mathbb{N}}$,
    (c) $\{n + \frac{1}{m}\}_{n,m \in \mathbb{N}}$.

22. Take a relation $\leq$ that is reflexive and transitive but not necessarily anti-symmetric. Define a new relation $\sim$ by setting $x \sim y$ if $x \leq y$ and $y \leq x$. Prove that $\sim$ is an equivalence relation. Moreover, show that $\leq$ induces a partial order on the set of equivalence classes.

Sets

23. State the Cantor-Schröder-Bernstein Theorem, and use it to prove that the intervals $(0,1)$ and $[0,1]$ have the same cardinality.

24. "There is no set of maximal cardinality."

    a) Re-write this sentence without using the word "maximal" nor the word "cardinality". (Hint: use quantifiers).

    b) Prove the sentence; state any theorems that you are using.

25. Which of the following sets have the same cardinality? Which have bigger cardinality than which? Justify your answer briefly. (Hint: rely on facts and theorems from the handout on sets).

    - The set of natural numbers;
    - The set of finite subsets of the natural numbers;
    - The set of finite subsets of the real numbers;
    - The set of arbitrary subsets of the natural numbers;
    - The set of arbitrary subsets of real numbers;
    - The set of algebraic numbers;
    - The set of rational numbers;
    - The set of rational numbers that are less than 5;
    - The set of real numbers;
    - The set of pairs of real numbers;
    - The set of polynomials with integer coefficients;
    - The set of polynomials with real coefficients;
    - The set of polynomials with complex coefficients.

26. Let A, B be nonempty sets.

    a) Suppose that there is an injective function $f\colon A \to B$. Show that there is a surjective function $g\colon B \to A$.

    b) Suppose that there is a surjective function $g\colon B \to A$. Show that there is an injective function $f\colon A \to B$.

27. Let A and B be sets.
    Prove that "there exists a bijection $f\colon A \to B$" is an equivalence relation.
    Denote the equivalence class of A by $|A|$.
    Prove that the relation $|A| \leq |B|$ given by "there exists an injection $f\colon A \to B$" is well defined.
    Prove that this relation is transitive.

Graph Colourings

28. Define the Ramsey number $R(s,t)$, where $s,t \geq 2$ are integers.

29. Prove that $R(s,2) = s$ for all $s \geq 2$.

30. Give an example that shows that $R(3,3) > 5$.

31. Prove that $R(s,t) \leq R(s,t-1) + R(s-1,t)$.

Straightedge and compass constructions; constructible numbers

32. Given a segment of length 4, give a straightedge and compass construction that produces a segment of length $\sqrt{3}$. You may use any of the following constructions as single steps: perpendicular bisector of a segment, parallel or perpendicular to a given line through a given point, angle bisector.

33. Use the definition of "constructible number" to show that, for a positive number x, if x is constructible then $\sqrt{x}$ is also constructible.

34. Let x be a positive number. Suppose that x is not constructible. Prove that $\sqrt{x}$ is not constructible.

35. Show that, given two segments of lengths a and b and a rational number r, there is a straightedge-and-compass construction of a segment of length (i) $a+b$, (ii) $a-b$ (assuming $a > b$), (iii) ra.
    Show that, if we are also given a segment of length 1, then there is a straightedge-and-compass construction of a segment of length (iv) a/b, (v) ab, (vi) $\sqrt{a}$.

    If the construction relies on a simpler construction, say what this simpler construction does and where it is used. If the proof that the construction works uses some fact about triangles, state this fact and say where it is used.

36. Use the algebraic criterion for a number to be constructible to determine which of the following numbers is constructible. (i) $2.1112323232323\ldots$, (ii) $1/(1+\sqrt{13})$, (iii) $\sqrt[4]{10}$, (iv) $\sqrt[6]{3}$.

37. Use the algebraic criterion for a number to be constructible to show that $\frac{\sqrt{2}}{2}$, $\frac{\sqrt{2+\sqrt{2}}}{2}$, $\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}$ are all constructible.

38. Let $F = \{\alpha + \beta\sqrt{3} \mid \alpha, \beta \in \mathbb{Q}\}$. Show that if (a_1, b_1) and (a_2, b_2) are distinct points in $\mathbb{R}^2$ whose coordinates a_1, b_1, a_2, b_2 all lie in F then the line that connects them can be given by an equation $\alpha x + \beta y + \gamma = 0$ where $\alpha, \beta, \gamma \in F$.

The isoperimetric inequality

39. Sketch a proof that the following two statements are equivalent without proving either one of these statements.
    - Of all planar shapes of the same perimeter, the circle has the largest area.
    - Of all planar shapes of the same area, the circle has the smallest perimeter.

40. Sketch a proof that the following two statements are equivalent without proving either one of these statements.
    - Of all boxes with a given surface area, the cube has the maximal volume.
    - Of all boxes with a given volume, the cube has the minimal surface area.

41. Derive an *exact* expression for the perimeter of each of the following shapes of area 1: a circle; a quadrant; a 3:2 rectangle; an equilateral triangle. Comment on these numbers in connection with the isoperimetric inequality.

42. For "reasonable" figures, write an expression "IQ" in terms of the area A and perimeter L that has the following properties.
    (i) If we rescale the figure, its IQ does not change
    (ii) If we increase A while fixing L, then the IQ increases.

    Show that, if we increase L while fixing A, then the IQ decreases. What can you say if only one of the two properties is true?

43. Do the area and perimeter determine the shape?