FUNDAMENTAL CONCEPTS IN DIFFERENTIAL GEOMETRY
FALL 2000
EXERCISES HANDOUT # 14

1. Let M^n be a manifold. Show that the following definitions of orientability are equivalent.

(a) There exists a never vanishing n-form on M.

(b) There exists an atlas $\{U_\alpha, \varphi_\alpha\}$ for M, such that if $U_\alpha \cap U_\beta \neq \emptyset$, then $\det \varphi_\alpha \varphi_\beta^{-1} > 0$.

2. Show that TM is always orientable.

3. (a) Show that if M and N are orientable, then so is $M \times N$.

(b) Show that if $M \times N$ and M are orientable then so is N.

4. Show that S^n is orientable.

5. Show that if M^n is a compact connected closed orientable manifold, then

$$H^n_{DR}(M) \neq 0.$$

6. (a) Let $R_i : S^{n-1} \to S^{n-1}$ be the map

$$R_i : (x^1, \dots, x^i, \dots, x^n) \mapsto (x^1, \dots, -x^i, \dots, x^n).$$

Compute $R_i^* : H^{n-1}_{DR}(S^{n-1}) \to H^{n-1}_{DR}(S^{n-1})$. Compute also the induced map $A^* : H^{n-1}_{DR}(S^{n-1}) \to H^{n-1}_{DR}(S^{n-1})$ for the antipodal map A.

(b) Show that $\mathbb{R}P^n$ is not orientable if n is even (use the projection map $F : S^n \to \mathbb{R}P^n$.)

7. (a) Show that $H^1_{DR}(S^1) \cong \mathbb{R}$ via the $\omega \mapsto \int_{S^1} \omega$ homomorphism.

(b)* Prove that if I is the unit interval M is a compact manifoldand and $i_0, i_1 : M \to M \times I$ are the obvious inclusions, then $i_0^* = i_1^*$. Use this to show that if f and g are homotopic maps $M \to N$, then $f^* = g^*$ and that i_0^*, i_1^* are isomorphisms. (Hint: Use what I have done in class to construct a suitable chain homotopy).

(c) Show that $H^2_{DR}(S^2) \cong \mathbb{R}$ via $\omega \mapsto \int_{S^2} \omega$. (Hint: Try to split the sphere into the Northern and Southern hemispheres and use Poincré's Lemma.)

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