Dror Bar-Natan: Talks: Zurich-080513

Local Khovanov Homology (1)

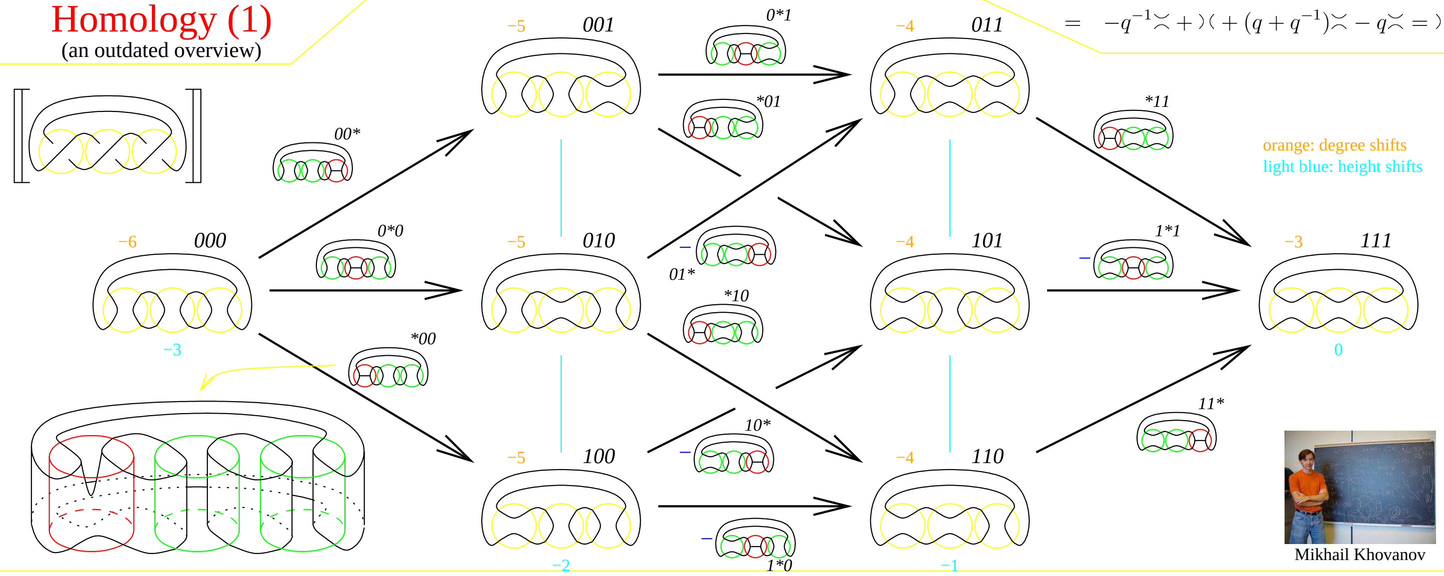
(an outdated overview)

The Jones polynomial: $\bigcirc = q + q^{-1}$

$\hat{J}: \times \mapsto q)(-q^2\asymp$, $\quad \hat{J}: \times \mapsto -q^{-2}\asymp + q^{-1})($,

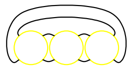
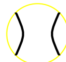
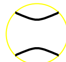
R2 $\hat{J}: \mapsto -q^{-1} + + - q$

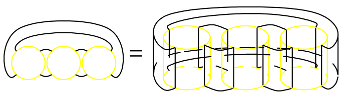
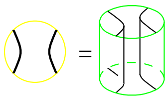
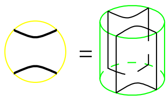
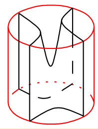
$= -q^{-1}\asymp +)(+ (q+q^{-1})\asymp - q\asymp =)($.



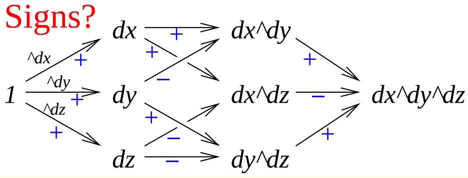
Mikhail Khovanov

What is it?

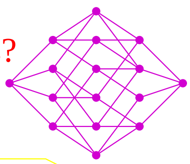
A cube for each knot/link projection; Vertices: All fillings of  with  or with .

Edges: All fillings of $I\times$  with $I\times$  or with $I\times$  and precisely one .

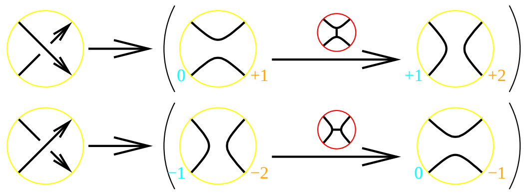
Signs?



More crossings?



General Crossings



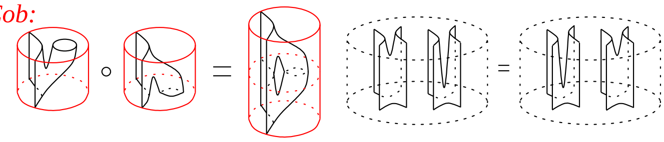
Where does it live?

In *Kom(Mat(<Cob> / {S, T, G, NC})) / homotopy*

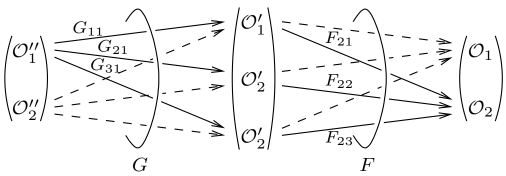
Kom: Complexes Mat: Matrices

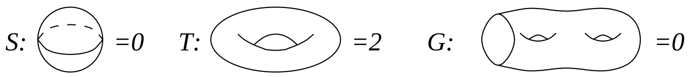
Cob: Cobordisms <...>: Formal lin. comb.

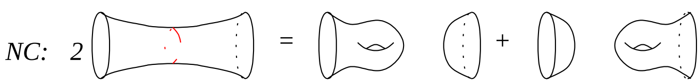
Cob:



Mat(C):







Complexes:

$$\Omega = \left(\Omega^{-n_-} \longrightarrow \Omega^{-n_-+1} \longrightarrow \cdots \qquad \longrightarrow \Omega^{n_+} \right)$$

Morphisms:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \Omega_0^{r-1} & \xrightarrow{d^{r-1}} & \Omega_0^r & \xrightarrow{d^r} & \Omega_0^{r+1} & \longrightarrow & \cdots \\ & & \downarrow F^{r-1} & & \downarrow F^r & & \downarrow F^{r+1} & & \\ \cdots & \longrightarrow & \Omega_1^{r-1} & \xrightarrow{d^{r-1}} & \Omega_1^r & \xrightarrow{d^r} & \Omega_1^{r+1} & \longrightarrow & \cdots \end{array}$$

Homotopies:

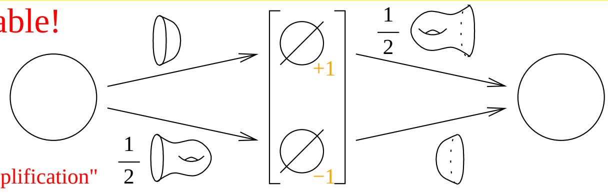
$$\begin{array}{ccccc} \Omega_0^{r-1} & \xrightarrow{d^{r-1}} & \Omega_0^r & \xrightarrow{d^r} & \Omega_0^{r+1} \\ F^{r-1}\Downarrow G^{r-1} & \swarrow h^r & F^r \Downarrow G^r & \swarrow h^{r+1} & F^{r+1}\Downarrow G^{r+1} \\ \Omega_1^{r-1} & \xrightarrow{d^{r-1}} & \Omega_1^r & \xrightarrow{d^r} & \Omega_1^{r+1} \end{array}$$

$$F^r - G^r = h^{r+1}d^r + d^{r-1}h^r$$

All arrows in an arbitrary additive category!

Computable!

via "complex simplification"



The Main Point. "The cube", $\mathrm{Kh}(L)$, is an up-to-homotopy invariant of knots and links. It's Euler characteristic is the Jones polynomial, yet it is strictly stronger than the Jones polynomial. It is functorial (in the appropriate sense) and practically computable.

The Categorification Speculative Paradigm. • Every object in math is the Euler characteristic of a complex.
• Every operation lifts to an operation between complexes.
• Every identity remains true, up to homotopy.

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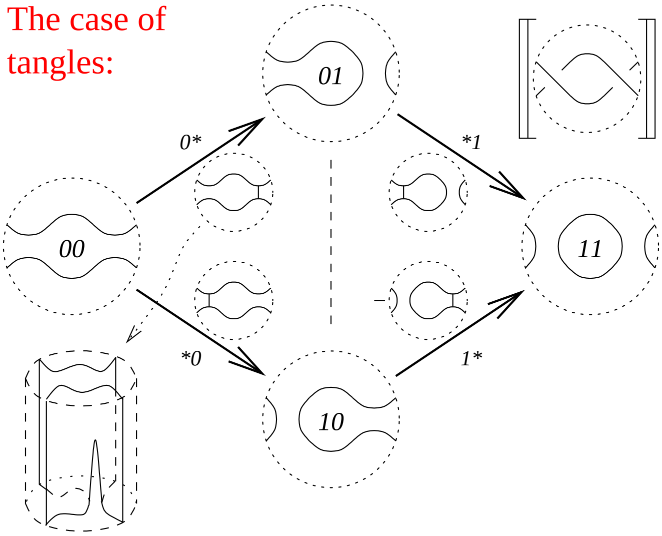
Local Khovanov Homology (2)

The case of tangles:



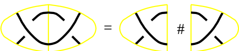
The Reduction Lemma. If ϕ is an isomorphism then the complex

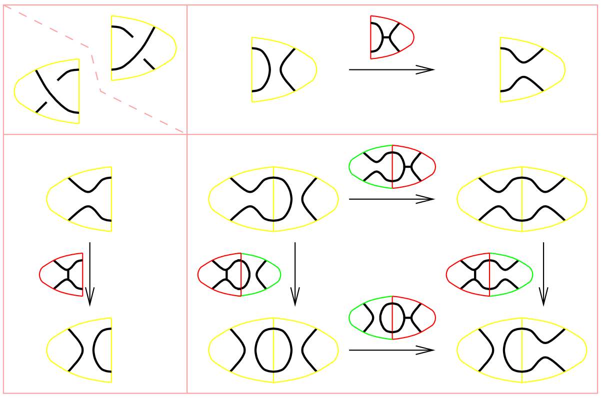
$$[C] \xrightarrow{\begin{pmatrix}\alpha\\ \beta\end{pmatrix}} \begin{bmatrix} b_1 \\ D \end{bmatrix} \xrightarrow{\begin{pmatrix}\phi & \delta\\ \gamma & \epsilon\end{pmatrix}} \begin{bmatrix} b_2 \\ E \end{bmatrix} \xrightarrow{\begin{pmatrix}\mu & \nu\end{pmatrix}} [F]$$

is isomorphic to the (direct sum) complex

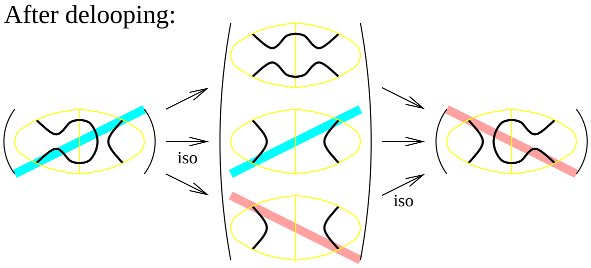
$$[C] \xrightarrow{\begin{pmatrix}0\\ \beta\end{pmatrix}} \begin{bmatrix} b_1 \\ D \end{bmatrix} \xrightarrow{\begin{pmatrix}\phi & 0\\ 0 & \epsilon-\gamma\phi^{-1}\delta\end{pmatrix}} \begin{bmatrix} b_2 \\ E \end{bmatrix} \xrightarrow{\begin{pmatrix}0 & \nu\end{pmatrix}} [F]$$

Invariance under R2.



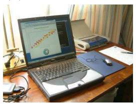


After delooping:



Kurt Reidemeister

Kh(T(7,6)).

In 1 day  says 

Old techniques:

~1,000 years,

~1GGb RAM.

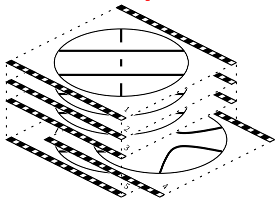
$\dim_j H_r$ is given by:

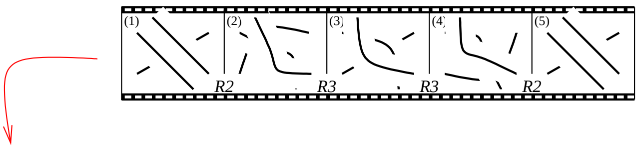
$j \setminus r$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
57																	1			1
55																	1	1		
53															1	2		1	1	
51													1	1		2	1			
49														3	1		1			
47												3	1		1					
45										2		1	2							
43								1		1	2									
41						1		1	2		1									
39						1	1		1											
37				1	1		1													
35					1															
33			1																	
31	1																			
29	1																			

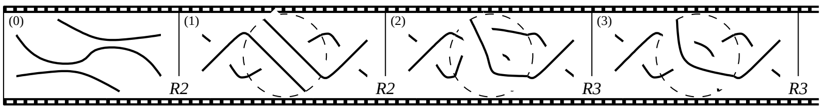
the alternating knots zone

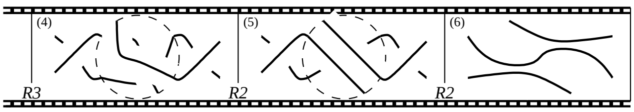
Functoriality / cobordisms.

M. Jacobsson



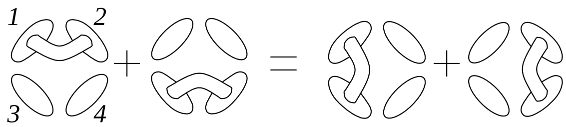






J. Rasmussen: This leads to a no−analysis proof of Milnor's conjecture

A more general theory: Remove G and NC, add

4Tu: 

(minor further revisions are necessary)

"God created the knots,

all else in topology is the work of mortals"

Leopold Kronecker (paraphrased)

Visit!

Edit!

http://katlas.org

http://www.math.toronto.edu/~drorbn/papers/Cobordism/

http://www.math.toronto.edu/~drorbn/papers/FastKh/

http://www.math.toronto.edu/~drorbn/Talks/Zurich−080513/