Homomorphic Expansions and w−Knots

Dror Bar−Natan, UWO February 2010,
http://www.math.toronto.edu/~drorbn/Talks/UWO−100225/

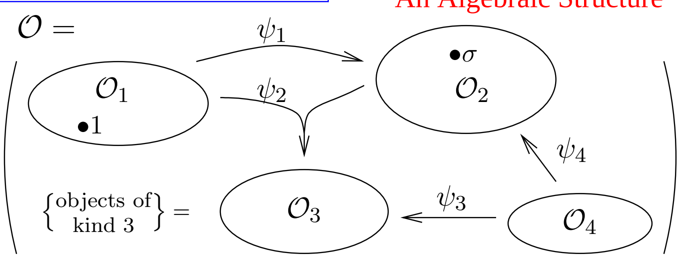
Abstract Even though little known, the notion of a "homomorphic expansion" is extremely general; it makes sense in the context of practically *any* algebraic structure, be it a group, or a group homomorphism, or a quandle, or a planar algebra, or a circuit algebra with unzip operations, or whatever.

Even though little known, w-knots make a cool generalization of ordinary knots. They contain ordinary knots and are contained in 2-knots in 4-space and are easier than the latter. They are a quotient of "virtual knots" and are easier then those.

My talk will be about these two notions, homomorphic expansions and w-knots, and about what happens when the two are put together. Lie algebras arise, and Lie groups, and the Kashiwara-Vergne statement, which is one of the deeper statements about the relationship between Lie groups and Lie algebras.

There are also u-knots, and v-knots, and f-knots, and other things which are not knots at all, and there are equally nifty things to say about homomorphic expansions for all those. But not today.

"An Algebraic Structure"



- Has kinds, objects, operations, and maybe constants.
- Perhaps subject to some axioms.
- We always allow formal linear combinations.

Homomorphic expansions for a filtered algebraic structure $\mathcal{K}$:

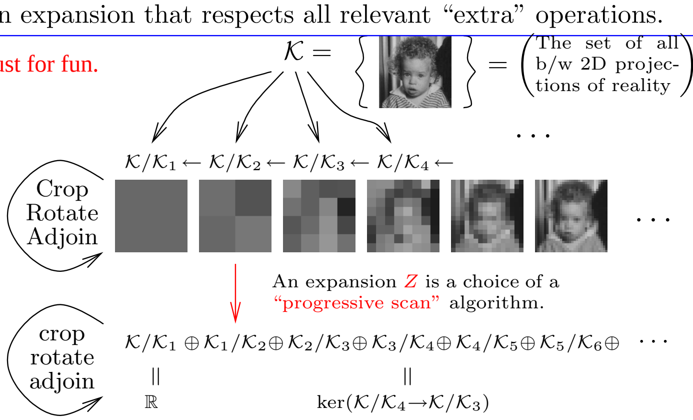
$$\text{ops} \circlearrowleft \mathcal{K} = \mathcal{K}_0 \supset \mathcal{K}_1 \supset \mathcal{K}_2 \supset \mathcal{K}_3 \supset \dots$$

$$\Downarrow \qquad \downarrow Z$$

$$\text{ops} \circlearrowleft \operatorname{gr}\mathcal{K} := \mathcal{K}_0/\mathcal{K}_1 \oplus \mathcal{K}_1/\mathcal{K}_2 \oplus \mathcal{K}_2/\mathcal{K}_3 \oplus \mathcal{K}_3/\mathcal{K}_4 \oplus \dots$$

An **expansion** is a filtration respecting $Z : \mathcal{K} \to \operatorname{gr}\mathcal{K}$ that "covers" the identity on $\operatorname{gr}\mathcal{K}$. A **homomorphic expansion** is an expansion that respects all relevant "extra" operations.

Just for fun.



Filtered algebraic structures are cheap and plenty. In any $\mathcal{K}$, allow formal linear combinations, let $\mathcal{K}_1$ be the ideal generated by differences (the "augmentation ideal"), and let $\mathcal{K}_m := \langle (\mathcal{K}_1)^m \rangle$ (using all available "products").

[1] http://qlink.queensu.ca/~4lb11/interesting.html 25/2/10, 2:52pm
Also see http://www.math.toronto.edu/~drorbn/papers/WKO/

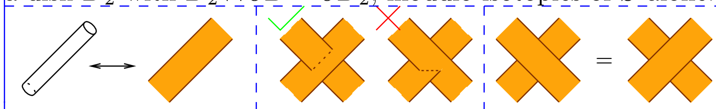
Examples. 1. The projectivization of a group is a graded associative algebra. 2. Quandle: a set Q with an op $\wedge$ s.t.

$$1 \wedge x = 1, \quad x \wedge 1 = x \wedge x = x, \qquad \text{(appetizers)}$$
$$(x \wedge y) \wedge z = (x \wedge z) \wedge (y \wedge z). \qquad \text{(main)}$$

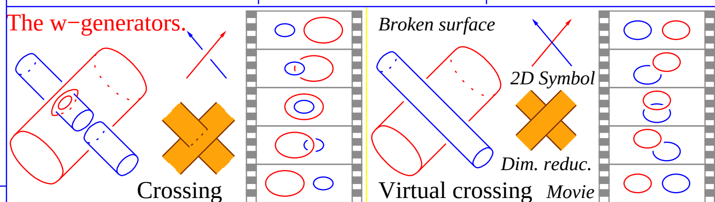
$\operatorname{proj} Q$ is a graded Lie algebra: set $\bar{v} := (v-1)$ (these generate I!), feed $1+\bar{x}$, $1+\bar{y}$, $1+\bar{z}$ in (main), collect the surviving terms of lowest degree:

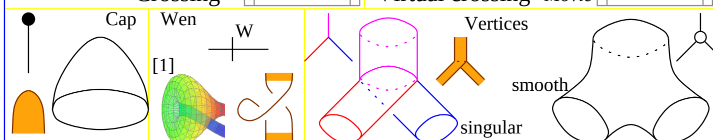
$$(\bar{x} \wedge \bar{y}) \wedge \bar{z} = (\bar{x} \wedge \bar{z}) \wedge \bar{y} + \bar{x} \wedge (\bar{y} \wedge \bar{z}).$$

A Ribbon 2-Knot is a surface S embedded in $\mathbb{R}^4$ that bounds an immersed handlebody B, with only "ribbon singularities"; a ribbon singularity is a disk D of trasverse double points, whose preimages in B are a disk D_1 in the interior of B and a disk D_2 with $D_2 \cap \partial B = \partial D_2$, modulo isotopies of S alone.

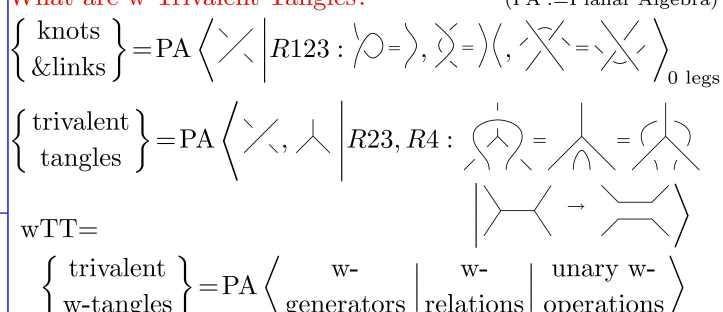


The w−generators.

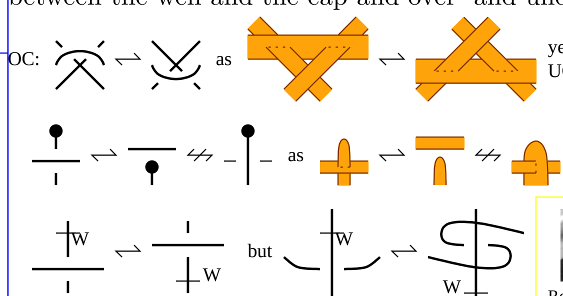




What are w-Trivalent Tangles? (PA :=Planar Algebra)



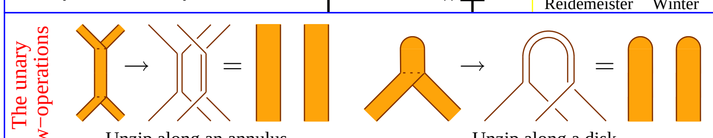
The w-relations include R234, VR1234, M, Overcrossings Commute (OC) but not UC, $W^2 = 1$, and funny interactions between the wen and the cap and over- and under-crossings:



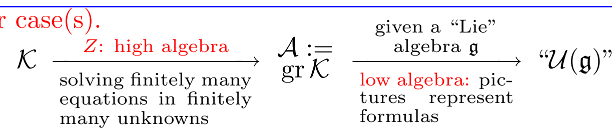
Challenge. Do the Reidemeister!

Reidemeister Winter

The unary w−operations



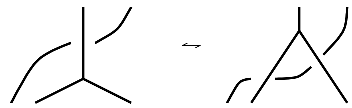
Our case(s).

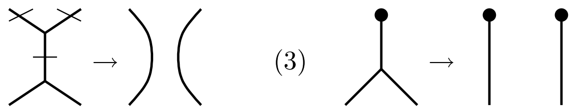


$\mathcal{K}$ is knot theory or **topology**; $\operatorname{gr}\mathcal{K}$ is finite **combinatorics**: bounded-complexity diagrams modulo simple relations.

Homomorphic Expansions and w−Knots, Page 2

Knot-Theoretic statement. There exists a homomorphic expansion Z for trivalent w-tangles. In particular, Z should respect $R4$ and intertwine annulus and disk unzips:

(1) 

(2) 

Convolutions statement (Kashiwara-Vergne). Convolutions of invariant functions on a Lie group agree with convolutions of invariant functions on its Lie algebra. More accurately, let G be a finite dimensional Lie group and let $\mathfrak{g}$ be its Lie algebra, let $j:\mathfrak{g}\to\mathbb{R}$ be the Jacobian of the exponential map $\exp:\mathfrak{g}\to G$, and let $\Phi:\text{Fun}(G)\to\text{Fun}(\mathfrak{g})$ be given by $\Phi(f)(x):=j^{1/2}(x)f(\exp x)$. Then if $f,g\in\text{Fun}(G)$ are Ad-invariant and supported near the identity, then

$$\Phi(f)\star\Phi(g)=\Phi(f\star g).$$


Kashiwara

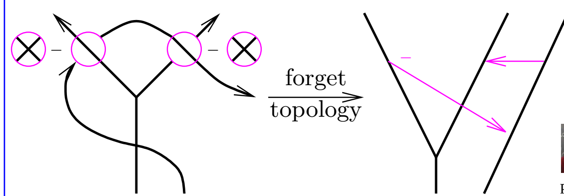

Vergne


Alekseev


Torossian

Measure theoretic statement. Ignoring all j's, there exists a measure preserving and orbit preserving transformation $T:\mathfrak{g}_x\times\mathfrak{g}_y\to\mathfrak{g}_x\times\mathfrak{g}_y$ for which $e^{x+y}\circ T=e^xe^y$.

Top. to Comb. $\text{gr}_m\,\text{wTT}:=\{m-\text{cubes}\}/\{(m+1)-\text{cubes}\}$:

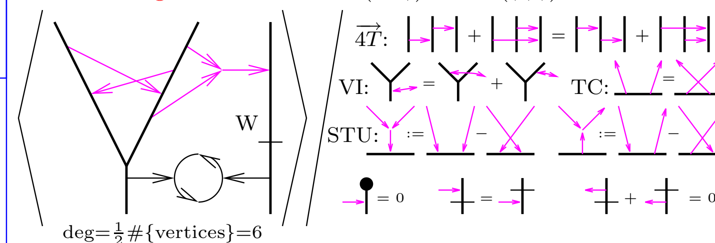



Vassiliev

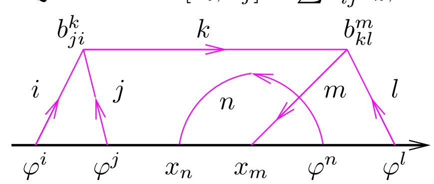

Polyak


Goussarov

w-Jacobi diagrams and $\mathcal{A}$. $\mathcal{A}^w(Y\uparrow)\cong\mathcal{A}^w(\uparrow\uparrow\uparrow)$ is



Diagrammatic to Algebraic. With (x_i) and (φ^j) dual bases of $\mathfrak{g}$ and $\mathfrak{g}^*$ and with $[x_i,x_j]=\sum b_{ij}^k x_k$, we have $\mathcal{A}^w\to\mathcal{U}$ via



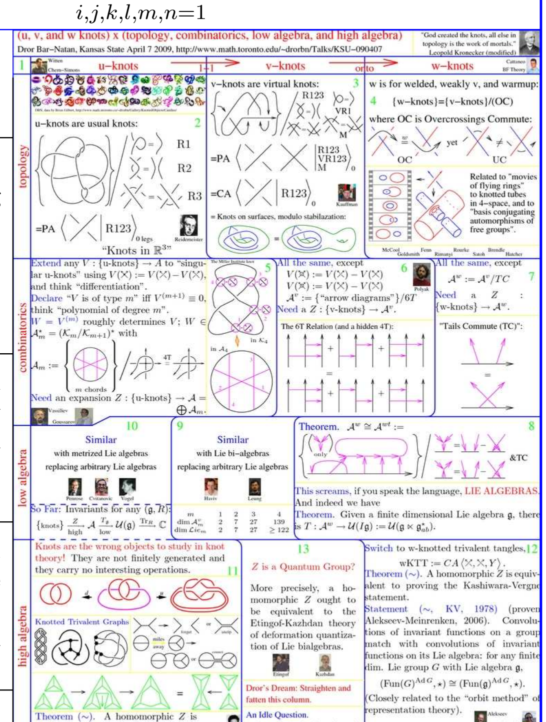

Penrose


Cvitanovic

$$\longrightarrow\sum_{i,j,k,l,m,n=1}^{\dim\mathfrak{g}}b_{ij}^kb_{kl}^m\varphi^i\varphi^jx_nx_m\varphi^l\in\mathcal{U}(I\mathfrak{g})$$

The u−v−w Story

	u-Knots ⟶	v-Knots ⟶	w-Knots
Topology	Ordinary (usual) knotted objects in 3D — braids, knots, links, tangles, knotted graphs, etc.	Virtual knotted objects — "algebraic" knotted objects, or "not specifically embedded" knotted objects; knots drawn on a surface, modulo stabilization.	Ribbon knotted objects in 4D; "flying rings". Like v, but also with "overcrossings commute".
Combinatorics	Chord diagrams and Jacobi diagrams, modulo $4T$, STU, IHX, etc.	Arrow diagrams and v-Jacobi diagrams, modulo $6T$ and various "directed" STUs and IHXs, etc.	Like v, but also with "tails commute". Only "two in one out" internal vertices.
Low Algebra	Finite dimensional metrized Lie algebras, representations, and associated spaces.	Finite dimensional Lie bi-algebras, representations, and associated spaces.	Finite dimensional co-commutative Lie bi-algebras (i.e., $\mathfrak{g}\ltimes\mathfrak{g}^*$), representations, and associated spaces.
High Algebra	The Drinfel'd theory of associators.	Likely, quantum groups and the Etingof-Kazhdan theory of quantization of Lie bi-algebras.	The Kashiwara-Vergne-Alekseev-Torossian theory of convolutions on Lie groups and Lie algebras.

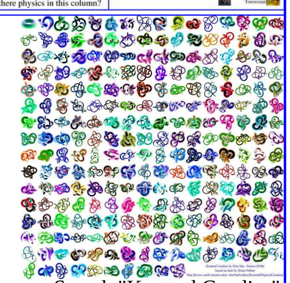


Some Propaganda

"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org


w−Search "Knotted Candies"