

# Convolutions on Lie Groups and Lie Algebras and Ribbon 2-Knots

Dror Bar-Natan, Trieste May 2009, <http://www.math.toronto.edu/~drorbn/Talks/Trieste-0905>

**Disclaimer:**  
Rough edges remain!

"God created the knots, all else in topology is the work of mortals."  
Leopold Kronecker (modified)



**Convolutions statement** (Kashiwara-Vergne). Convolutions of invariant functions on a Lie group agree with convolutions of invariant functions on its Lie algebra. More accurately, let  $G$  be a finite dimensional Lie group and let  $\mathfrak{g}$  be its Lie algebra, let  $j : \mathfrak{g} \rightarrow \mathbb{R}$  be the Jacobian of the exponential map  $\exp : \mathfrak{g} \rightarrow G$ , and let  $\Phi : \text{Fun}(G) \rightarrow \text{Fun}(\mathfrak{g})$  be given by  $\Phi(f)(x) := j^{1/2}(x)f(\exp x)$ . Then if  $f, g \in \text{Fun}(G)$  are Ad-invariant and supported near the identity, then

$$\Phi(f) \star \Phi(g) = \Phi(f \star g).$$

**Group-Ring statement.** There exists  $\omega^2 \in \text{Fun}(\mathfrak{g})^G$  so that for every  $\phi, \psi \in \text{Fun}(\mathfrak{g})^G$  (with small support), the following holds in  $\hat{U}(\mathfrak{g})$ :

$$\iint_{\mathfrak{g} \times \mathfrak{g}} \phi(x)\psi(y)\omega_{x+y}^2 e^{x+y} = \iint_{\mathfrak{g} \times \mathfrak{g}} \phi(x)\psi(y)\omega_x^2 \omega_y^2 e^x e^y. \quad (\text{shhh, } \omega^2 = j^{1/2})$$

(shhh, this is Dufo)

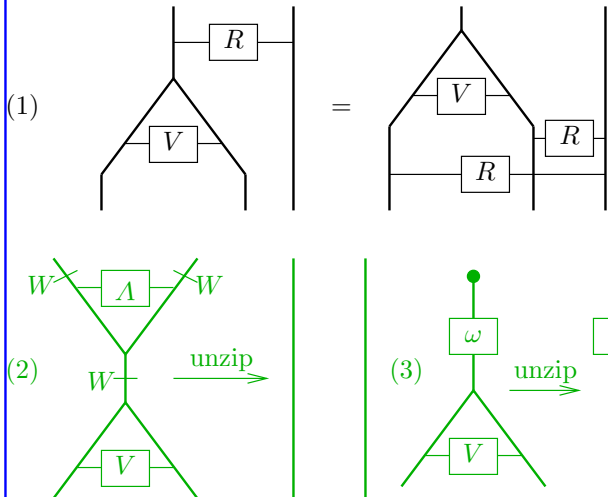
**Unitary statement.** There exists  $\omega \in \text{Fun}(\mathfrak{g})^G$  and an (infinite order) tangential differential operator  $V$  defined on  $\text{Fun}(\mathfrak{g}_x \times \mathfrak{g}_y)$  so that

- (1)  $V e^{\widehat{x+y}} = \widehat{e^x e^y} V$  (allowing  $\hat{U}(\mathfrak{g})$ -valued functions)
- (2)  $V V^* = I$       (3)  $V \omega_{x+y} = \omega_x \omega_y$

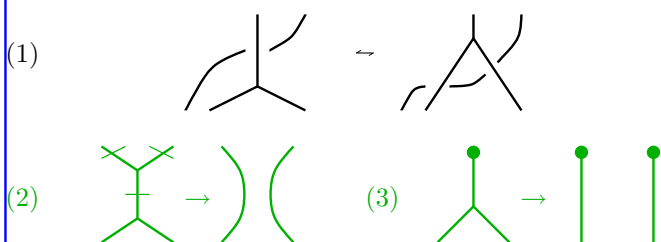
**Algebraic statement.** With  $I_{\mathfrak{g}} := \mathfrak{g}^* \rtimes \mathfrak{g}$ , with  $c : \hat{U}(I_{\mathfrak{g}}) \rightarrow \hat{U}(\mathfrak{g})/\hat{U}(\mathfrak{g}) = \hat{S}(\mathfrak{g}^*)$  the obvious projection, with  $S$  the antipode of  $\hat{U}(I_{\mathfrak{g}})$ , with  $W$  the automorphism of  $\hat{U}(I_{\mathfrak{g}})$  induced by flipping the sign of  $\mathfrak{g}^*$ , with  $r \in \mathfrak{g}^* \otimes \mathfrak{g}$  the identity element and with  $R = e^r \in \hat{U}(I_{\mathfrak{g}}) \otimes \hat{U}(\mathfrak{g})$  there exist  $\omega \in \hat{S}(\mathfrak{g}^*)$  and  $V \in \hat{U}(I_{\mathfrak{g}})^{\otimes 2}$  so that

- (1)  $V(\Delta \otimes 1)(R) = R^{13} R^{23} V$  in  $\hat{U}(I_{\mathfrak{g}})^{\otimes 2} \otimes \hat{U}(\mathfrak{g})$
- (2)  $V \cdot S W V = 1$       (3)  $(c \otimes c)(V \Delta(\omega)) = \omega \otimes \omega$

**Diagrammatic statement.** Let  $R = \exp \uparrow \uparrow \in \mathcal{A}^w(\uparrow \uparrow)$ . There exist  $\omega \in \mathcal{A}^w(\uparrow)$  and  $V \in \mathcal{A}^w(\uparrow \uparrow)$  so that



**Knot-Theoretic statement.** There exists a homomorphic expansion  $Z$  for trivalent w-tangles. In particular,  $Z$  should respect  $R4$  and intertwine annulus and disk unzips:



**The Orbit Method.** By Fourier analysis, the characters of  $(\text{Fun}(\mathfrak{g})^G, \star)$  correspond to coadjoint orbits in  $\mathfrak{g}^*$ . By averaging representation matrices and using Schur's lemma to replace intertwiners by scalars, to every irreducible representation of  $G$  we can assign a character of  $(\text{Fun}(G)^G, \star)$ .



Kashiwara



Vergne

**Measure theoretic statement.** Ignoring all  $\omega$ 's, there exists a measure preserving and orbit preserving transformation  $T : \mathfrak{g}_x \times \mathfrak{g}_y \rightarrow \mathfrak{g}_x \times \mathfrak{g}_y$  for which  $e^{x+y} \circ T = e^x e^y$ .

**Free Lie statement** (Kashiwara-Vergne). There exist convergent Lie series  $F$  and  $G$  so that

$$x + y - \log e^y e^x = (1 - e^{-\text{ad } x})F + (e^{\text{ad } y} - 1)G$$

$$\text{tr}(\text{ad } x) \partial_x F + \text{tr}(\text{ad } y) \partial_y G =$$

$$\frac{1}{2} \text{tr} \left( \frac{\text{ad } x}{e^{\text{ad } x} - 1} + \frac{\text{ad } y}{e^{\text{ad } y} - 1} - \frac{\text{ad } z}{e^{\text{ad } z} - 1} - 1 \right)$$

**Alekseev-Torossian statement.** There is an element  $F \in \text{TAut}_2$  with

$$F(x + y) = \log e^x e^y$$



Alekseev

and  $j(F) \in \text{im } \tilde{\delta} \subset \text{tr}_2$ , where for  $a \in \text{tr}_1$ ,  $\tilde{\delta}(a) := a(x) + a(y) - a(\log e^x e^y)$ .



Torossian

**Convolutions and Group Rings** (ignoring all Jacobians). If  $G$  is finite,  $(\text{Fun}(G), \star) \cong (\mathbb{R}G, \cdot)$  via  $T : f \mapsto \sum f(a)\tau(a)$ . For Lie  $\mathfrak{g}$  and  $G$ ,

$$\begin{array}{ccc} (\mathfrak{g}, +) \ni x & \xrightarrow{\tau} & e^x \in \hat{S}(\mathfrak{g}) & \psi \in \text{Fun}(\mathfrak{g}) & \xrightarrow{T} & \hat{S}(\mathfrak{g}) \\ \downarrow \exp & & \downarrow \chi & \text{so} & \downarrow \Phi^{-1} & \downarrow \chi \\ (G, \cdot) \ni e^x & \xrightarrow{\tau} & e^x \in \hat{U}(\mathfrak{g}) & \text{Fun}(G) & \xrightarrow{T} & \hat{U}(\mathfrak{g}) \end{array}$$

with  $T\psi = \int \psi(x)e^x dx \in \hat{S}(\mathfrak{g})$  and  $T\Phi^{-1}\psi = \int \psi(x)e^x \in \hat{U}(\mathfrak{g})$ . Given  $\psi_i \in \text{Fun}(\mathfrak{g})$  compare  $\Phi^{-1}(\psi_1) \star \Phi^{-1}(\psi_2)$  and  $\Phi^{-1}(\psi_1 \star \psi_2)$  in  $\hat{U}(\mathfrak{g})$ : (shhh,  $T$  is a "Fourier transform")

$$\star \text{ in } G : \iint \psi_1(x)\psi_2(y)e^x e^y \quad \star \text{ in } \mathfrak{g} : \iint \psi_1(x)\psi_2(y)e^{x+y}$$

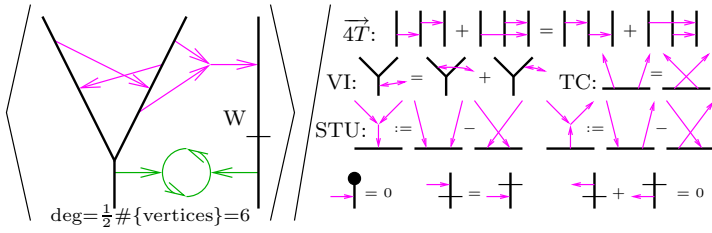
**Unitary  $\implies$  Group-Ring.**  $\iint \omega_{x+y}^2 e^{x+y} \phi(x)\psi(y)$

$$\begin{aligned} &= \langle \omega_{x+y}, \omega_{x+y} e^{x+y} \phi(x)\psi(y) \rangle = \langle V \omega_{x+y}, V e^{x+y} \phi(x)\psi(y) \omega_{x+y} \rangle \\ &= \langle \omega_x \omega_y, e^x e^y V \phi(x)\psi(y) \omega_{x+y} \rangle = \langle \omega_x \omega_y, e^x e^y \phi(x)\psi(y) \omega_x \omega_y \rangle \\ &= \iint \omega_x^2 \omega_y^2 e^x e^y \phi(x)\psi(y). \end{aligned}$$

**Unitary  $\iff$  Algebraic.** The key is to interpret  $\hat{U}(I_{\mathfrak{g}})$  as tangential differential operators on  $\text{Fun}(\mathfrak{g})$ :

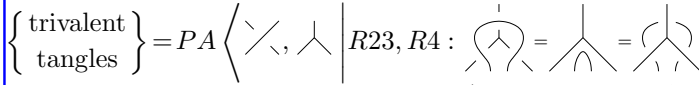
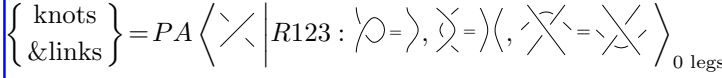
- $\varphi \in \mathfrak{g}^*$  becomes a multiplication operator.
- $x \in \mathfrak{g}$  becomes a tangential derivation, in the direction of the action of  $\text{ad } x$ :  $(x\varphi)(y) := \varphi([x, y])$ .
- $c$  is now "the constant term".

w-Jacobi diagrams and  $\mathcal{A}$ .  $\mathcal{A}^w(Y \uparrow) \cong \mathcal{A}^w(\uparrow \uparrow \uparrow)$  is

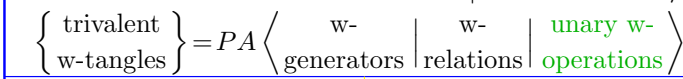


$\Delta$  acts by double and sum,  $S$  by reverse and negate.

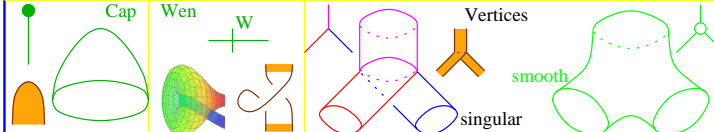
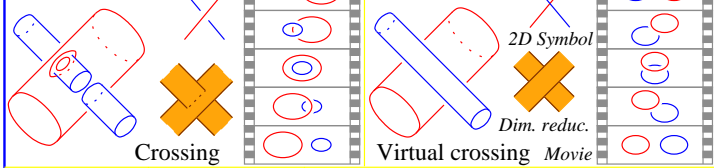
What are w-Trivalent Tangles?



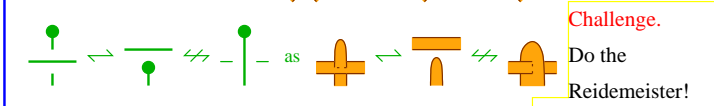
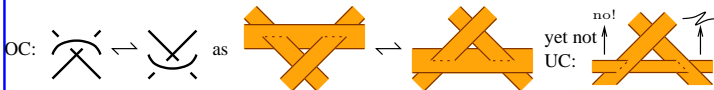
wTT =



The w-generators.



The w-relations include R234, VR1234, M, Overcrossings Commute (OC) but not UC,  $W^2 = 1$ , and funny interactions between the wen and the cap and over- and under-crossings:

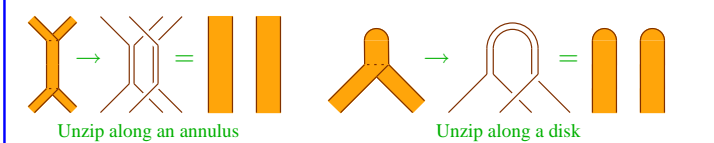


Challenge.

Do the Reidemeister!

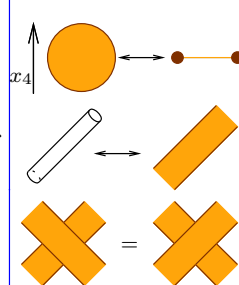


The unary w-operations.



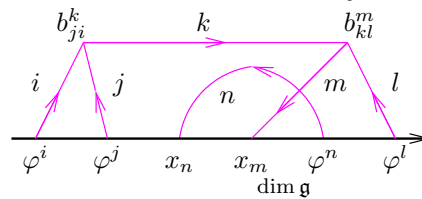
A Ribbon 2-Knot is a surface  $S$  embedded in  $\mathbb{R}^4$  that bounds an immersed handlebody  $B$ , with only "ribbon singularities"; a ribbon singularity is a disk  $D$  of transverse double points, whose preimages in  $B$  are a disk  $D_1$  in the interior of  $B$  and a disk  $D_2$  with  $D_2 \cap \partial B = \partial D_2$ , modulo isotopies of  $S$  alone.

Dimensional reduction



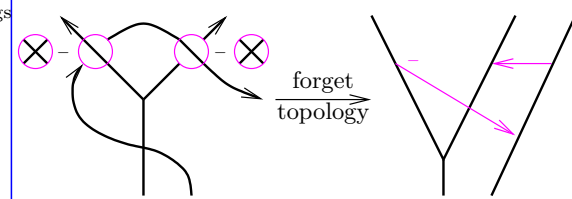
Example.

Diagrammatic to Algebraic. With  $(x_i)$  and  $(\varphi^j)$  dual bases of  $\mathfrak{g}$  and  $\mathfrak{g}^*$  and with  $[x_i, x_j] = \sum b_{ij}^k x_k$ , we have  $\mathcal{A}^w \rightarrow \mathcal{U}$  via



$$\rightarrow \sum_{i,j,k,l,m,n=1}^{\dim \mathfrak{g}} b_{ij}^k b_{kl}^m \varphi^i \varphi^j x_n x_m \varphi^l \in \mathcal{U}(\mathfrak{g})$$

From wTT to  $\mathcal{A}^w$ .  $\text{gr}_m \text{wTT} := \{m\text{-cubes}\} / \{(m+1)\text{-cubes}\}$ :



Homomorphic expansions for a filtered algebraic structure  $\mathcal{K}$ :

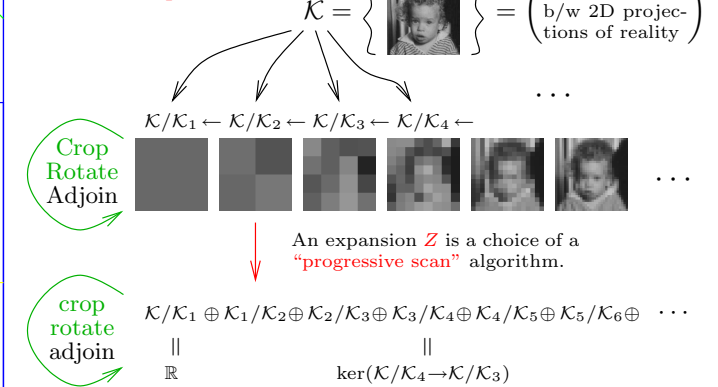
$$\text{ops} \subset \mathcal{K} = \mathcal{K}_0 \supset \mathcal{K}_1 \supset \mathcal{K}_2 \supset \mathcal{K}_3 \supset \dots$$

$$\downarrow \quad \downarrow Z$$

$$\text{ops} \subset \text{gr } \mathcal{K} := \mathcal{K}_0/\mathcal{K}_1 \oplus \mathcal{K}_1/\mathcal{K}_2 \oplus \mathcal{K}_2/\mathcal{K}_3 \oplus \mathcal{K}_3/\mathcal{K}_4 \oplus \dots$$

An expansion is a filtration respecting  $Z : \mathcal{K} \rightarrow \text{gr } \mathcal{K}$  that "covers" the identity on  $\text{gr } \mathcal{K}$ . A homomorphic expansion is an expansion that respects all relevant "extra" operations.

A concrete example.



Our case(s).

$$\mathcal{K} \xrightarrow[\text{solving finitely many equations in finitely many unknowns}]{Z: \text{high algebra}} \mathcal{A} := \text{gr } \mathcal{K} \xrightarrow[\text{low algebra: pictures represent formulas}]{\text{given a "Lie" algebra } \mathfrak{g}} \mathcal{U}(\mathfrak{g})$$

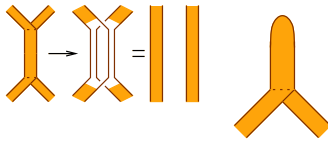
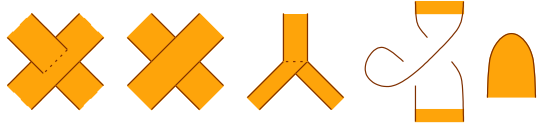
$\mathcal{K}$  is knot theory or topology;  $\text{gr } \mathcal{K}$  is finite combinatorics: bounded-complexity diagrams modulo simple relations.

But we have (at least) three knot theories,  $u \rightarrow v \rightarrow w$ , and thus their "high algebras" are related!

Filtered algebraic structures are cheap and plenty. In any  $\mathcal{K}$ , allow formal linear combinations, let  $\mathcal{K}_1$  be the ideal generated by differences (the "augmentation ideal"), and let  $\mathcal{K}_m := \langle (\mathcal{K}_1)^m \rangle$  (using all available "products").

We skipped... • The Alexander • v-Knots, quantum groups and polynomial and Milnor numbers. Etingof-Kazhdan. • u-Knots and Drinfel'd associators. • BF theory and the successful religion of path integrals.

Unitary  $\implies$  Group-Ring.  $\iint \omega_{x+y}^2 e^{x+y} \phi(x) \psi(y)$   
 $= \langle \omega_{x+y}, \omega_{x+y} e^{x+y} \phi(x) \psi(y) \rangle = \langle V \omega_{x+y}, V e^{x+y} \phi(x) \psi(y) \omega_{x+y} \rangle$   
 $= \langle \omega_x \omega_y, e^x e^y V \phi(x) \psi(y) \omega_{x+y} \rangle = \langle \omega_x \omega_y, e^x e^y \phi(x) \psi(y) \omega_x \omega_y \rangle$   
 $= \iint \omega_x^2 \omega_y^2 e^x e^y \phi(x) \psi(y).$



Draft