From Stonehenge to Witten – Some Further Details


Stonehenge

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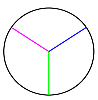
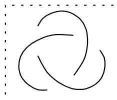
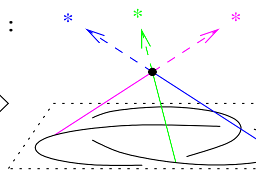
Witten

We the generating function of all stellar coincidences:

$$Z(K) := \lim_{N\to\infty} \sum_{3\text{-valent } D} \frac{1}{2^c c! \binom{N}{e}} \langle D, K\rangle_{\overline{\Pi}} D \cdot \begin{pmatrix}\text{framing-}\\ \text{dependent}\\ \text{counter-term}\end{pmatrix} \in \mathcal{A}(\circlearrowleft)$$

$$\langle D, K\rangle_{\overline{\Pi}} := \begin{pmatrix}\text{The signed Stonehenge}\\ \text{pairing of } D \text{ and } K\end{pmatrix} :$$

$D=$  $K=$  $\Longrightarrow$ 

count with signs

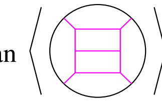
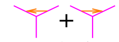
Theorem. Modulo Relations, $Z(K)$ is a knot invariant!

Dylan Thurston

N := # of stars

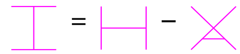
c := # of chopsticks

e := # of edges of D

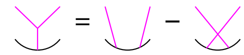
$\mathcal{A}(\circlearrowleft)$:=Span  / oriented vertices, AS:  =0, & more relations

When deforming, catastrophes occur when:

A plane moves over an intersection point – Solution: Impose IHX,

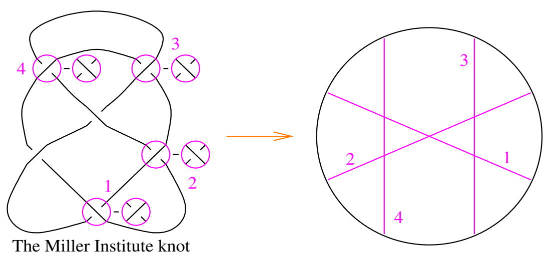


An intersection line cuts through the knot – Solution: Impose STU,



The Gauss curve slides over a star – Solution: Multiply by a framing–dependent counter–term.

$$\int_{\mathfrak{g}\text{-connections}} \mathcal{D}A\, hol_K(A) \exp\left[\frac{ik}{4\pi}\int_{\mathbb{R}^3} \operatorname{tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)\right] \longrightarrow \sum_{D:\ \text{Feynman diagram}} W_{\mathfrak{g}}(D) \sum\!\!\!\!\!\!\not\ \ \mathcal{E}(D) \longrightarrow \sum_{D:\ \text{Feynman diagram}} D \sum\!\!\!\!\!\!\not\ \ \mathcal{E}(D)$$



The Miller Institute knot

Definition. V is finite type (Vassiliev, Goussarov) if it vanishes on sufficiently large alternations as on the right

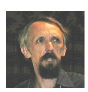
Theorem. All knot polynomials (Conway, Jones, etc.) are of finite type.


Goussarov

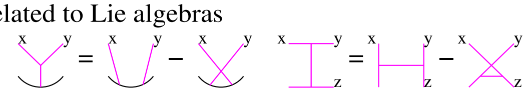
Conjecture. (Taylor's theorem) Finite type invariants separate knots.

Theorem. $Z(K)$ is a universal finite type invariant!

(sketch: to dance in many parties, you need many feet).


Vassiliev

Related to Lie algebras



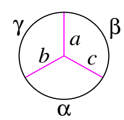
[x,y] = xy − yx [[x,y],z]=[x,[y,z]]−[y,[x,z]]


Sophus Lie

More precisely, let $\mathfrak{g} = \langle X_a\rangle$ be a Lie algebra with an orthonormal basis, and let $R = \langle v_\alpha\rangle$ be a representation. Set

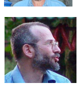
$$f_{abc} := \langle [a,b], c\rangle \qquad X_a v_\beta = \sum_\beta r^\beta_{a\gamma} v_\gamma$$

and then

$W_{\mathfrak{g},R}$:  $\longrightarrow \sum_{abc\alpha\beta\gamma} f_{abc} r^\beta_{a\gamma} r^\gamma_{b\alpha} r^\alpha_{c\beta}$

$W_{\mathfrak{g},R}\circ Z$ is often interesting:

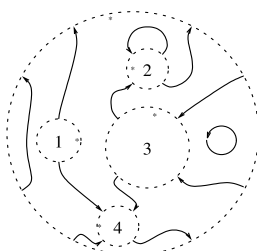
$\mathfrak{g} = sl(2)$ $\longrightarrow$  The Jones polynomial

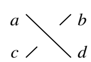
$\mathfrak{g} = sl(N)$ $\longrightarrow$  The HOMFLYPT polynomial

Przytycki

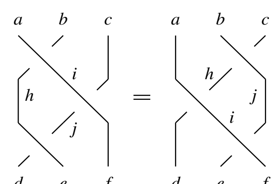
$\mathfrak{g} = so(N)$ $\longrightarrow$  The Kauffman polynomial

Planar algebra and the Yang–Baxter equation



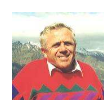


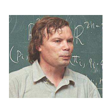
$$R^{ab}_{cd}$$

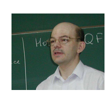


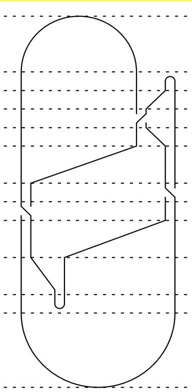
$$R^{ab}_{hi} R^{ic}_{jf} R^{hj}_{de} = R^{ah}_{di} R^{bc}_{hj} R^{ij}_{ef}$$


Yang

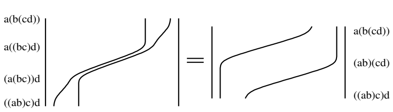

Baxter

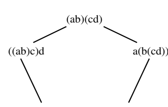

Reshetikhin

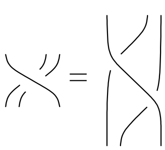

Turaev

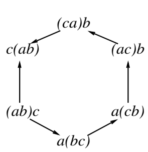


Parenthesized tangles, the pentagon and hexagon










Drinfel'd

Kauffman's bracket and the Jones polynomial

$\langle \times \rangle = \langle \asymp \rangle - q\langle)(\rangle$
0-smoothing 1-smoothing

$\langle \bigcirc^k \rangle = (q+q^{-1})^k$

$\hat{J}(L) = (-1)^{n_-} q^{n_+ - 2n_-} \langle L \rangle$

(n_+, n_-) count $(\nwarrow\!\!\!\!\nearrow, \nearrow\!\!\!\!\nwarrow)$

Claim $\hat{J}(\,) = \hat{J}()($

Indeed,

$\langle\ \rangle = \langle\ \rangle - q\langle\ \rangle$

$-q\langle\ \rangle + q^2\langle\ \rangle$

$(q+q^{-1})$

$= -q\langle)(\rangle$

"God created the knots, all else in topology is the work of man." This handout is at http://www.math.toronto.edu/~drorbn/Talks/Oporto–0407

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