

Dror Bar-Natan: Talks: Hanoi-0708: **Following Lin: Expansions for Groups**



Riverside, April 2000      Kyoto, September 2001  
See Lin's "Power Series Expansions and Invariants of Links", 1993 Georgia International Topology Conference, AMS/IP Studies in Adv. Math. **2** (1997) 184-202.

- Vaughan's Hierarchy** (generalized, unauthorized)
- ☺ Computation
  - ☺ Formula
  - ☺ Proof
  - ☺ Theory
  - ☺ Dream

**The Magnus and Exponential Expansions**

$$Z_{1,2} : G_n = \left( \begin{array}{c} \text{free group} \\ \text{on} \\ X_1, \dots, X_n \end{array} \right) \rightarrow \hat{A}_n = \left( \begin{array}{c} \text{completed free} \\ \text{associative} \\ \text{algebra on} \\ x_1, \dots, x_n \end{array} \right)$$

by  $X_i \mapsto 1 + x_i$  or  $e^{x_i}$   
 $X_i^{-1} \mapsto 1 - x_i + x_i^2 - \dots$  or  $e^{-x_i}$ .

**What's "An Expansion"?** A filtration-preserving isomorphism  $Z : C(G) \rightarrow \mathcal{A}(G)$  where

$$I := \{ \sum a_i g_i : \sum a_i = 0 \} \subset \mathbb{C}G$$

$$\mathbb{C}G = I^0 \supset I^1 \supset I^2 \supset I^3 \supset \dots$$

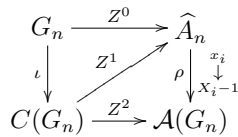
$$C(G) := \varprojlim_k \mathbb{C}G/I^k \rightarrow \dots \rightarrow \mathbb{C}G/I^2 \rightarrow \mathbb{C}G/I \rightarrow 0$$

is filtered by  $F_m C(G) := \varprojlim_{k>m} I^m/I^k$  and  $\mathcal{A}(G) := \text{gr } C(G) = \bigoplus I^m/I^{m+1}$ .  
 So all expansions are "equivalent"

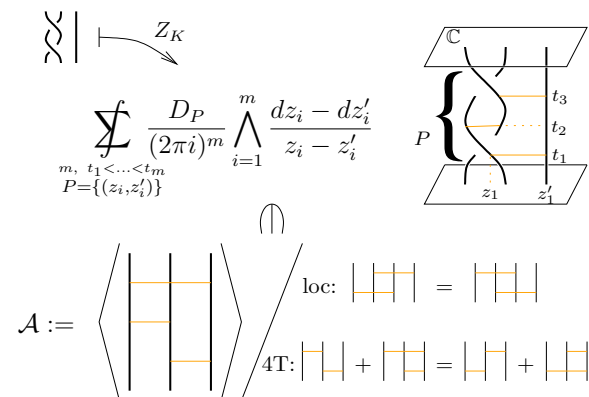
**Think duals!**  $C(G)^*$  are "finite type invariants".  
 $\mathcal{A}(G)^*$  are "weight systems".  
 $Z$  is a "universal finite type invariant".

$Z_{1,2}$  are Expansions. With  $Z^0 = Z_1$  or  $Z^0 = Z_2$ :

1.  $\iota$  is automatic.
  2.  $\rho$  is well-defined.
  3.  $Z^0|_{I^m} \subset F_m \mathcal{A}_n$ .
  4.  $Z^0$  descends to  $Z^1$ .
  5. Define  $Z^2$ .
  6.  $\rho$  is surjective.
  7.  $\text{gr } Z^2$  is the identity.
  8.  $Z^2$  is an isomorphism.
  9.  $\rho$  is an isomorphism.
- Everything generalizes, step 2 sometimes becomes tricky.



**The Kontsevich Integral for Braids**



M. Kontsevich

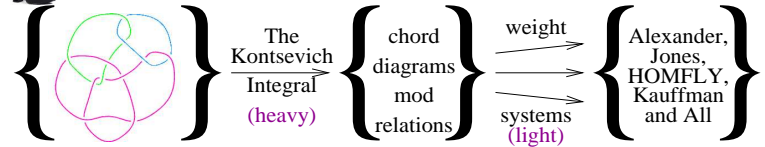
Which other groups / groupoids / categories have expansions?



**Dror's Dream / Obsession:**

"Unify" quantum groups – find one object that contains all.

**Example:** One invariant to rule them all:



Easy! Universal! A Morphism! Unique! An Isomorphism!

**What is a "Quantum Group"?** For now, a "deformation of the trivial" solution in  $\mathcal{U}(\mathfrak{g})^{\otimes*}[[\hbar]]$  of the major equations:

$$(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta \quad R^{-1}\Delta R = \Delta^{op}$$

$$(\Delta \otimes 1)R = R^{23}R^{13} \quad (1 \otimes \Delta)R = R^{12}R^{13}$$

(as well as a few minor equations).

**Dror's Guess:** A unified object exists; we'll need:

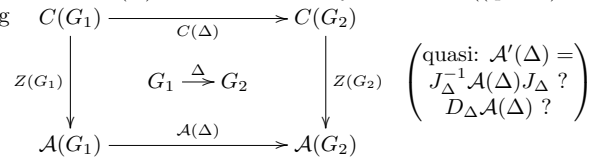
1. Expansions as in Lin / universal finite type invariants.
2. Naturality / functoriality.
3. Knotted graphs, especially trivalent.
4. Associators following Drinfel'd.
5. The work of Etingof and Kazhdan on bialgebras.
6. Virtual braids / knots / knotted graphs.
7. Polyak (LMP 54) & Haviv (arXiv:math/0211031) on arrow diagrams. (and when construction ends, we'll dump the scaffolding)

**Why care?**  
 Quantum groups computable invariants make!

**Visit!**  
 katlas.org  
**Edit!**

**(Quasi?) Natural Expansions**

$G \mapsto C(G)$  and  $G \mapsto \mathcal{A}(G)$  are functors. Can you choose a ((quasi?) natural)  $Z$  satisfying

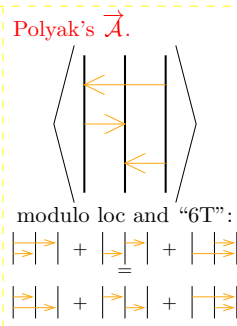
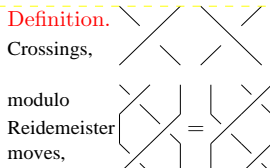


Perhaps just on a subcategory of **Groups**? Perhaps **Braids** with strands addition, deletion and doubling:



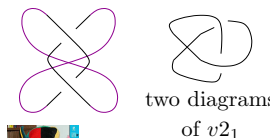
**Virtual Braids**

crossings are real, strands go virtual

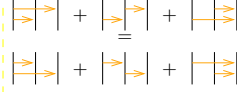


**Lie bialgebras.** The  $\mathfrak{g}$  in a sum  $\mathfrak{g} \oplus \mathfrak{g}^*$  which in itself is a Lie algebra with subalgebras  $\mathfrak{g}$  and  $\mathfrak{g}^*$ , and in which the tautological metric is invariant.

but the linkages between crossings are "virtual":



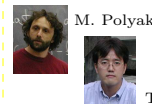
modulo loc and "6T":



**Why bother?** Their deformations are quantum groups, and their diagrammatic universalization is  $\vec{\mathcal{A}}$ .



L. Kauffman

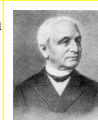


T. Ohtsuki



**Question** Can you interpret quantum groups as (quasi?)-natural expansions on virtual braids?

**Dror's Guess:** No, but the effort will be worthwhile.



"God created the knots, all else in topology is the work of mortals"

Leopold Kronecker (modified)