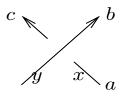
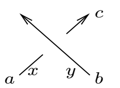
Dror Bar−Natan: Talks: Sandbjerg−0810:

# The Penultimate Alexander Invariant

## A Definition of the MVA (From [Ar])



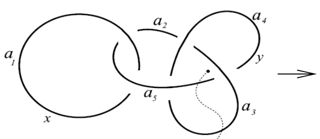

| | $a$ | $b$ | $c$ |
|---|---|---|---|
| $c$ | $-1$ | $1-x$ | $y$ |




| | $a$ | $b$ | $c$ |
|---|---|---|---|
| $c$ | $-y$ | $x-1$ | $1$ |

Joint with Jana Archibald

$$A=\frac{(-)^{i+j}\det(M_i^j)}{w_i(t_i-1)}\prod_k t_k^{\frac{\mathrm{rot}(k)-\mu(k)}{2}}$$

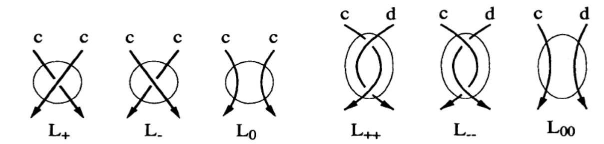


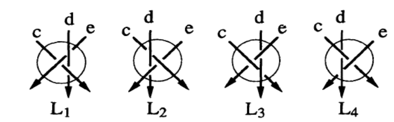

$\to$

| | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
|---|---|---|---|---|---|
| $a_1$ | $y-1$ | 0 | 0 | 0 | $1-x$ |
| $a_2$ | $1-y$ | $x$ | 0 | 0 | $-1$ |
| $a_3$ | $-y$ | 0 | 1 | $y-1$ | 0 |
| $a_4$ | 0 | 0 | $-y$ | 1 | $y-1$ |
| $a_5$ | 0 | 0 | $y-1$ | $-y$ | 1 |

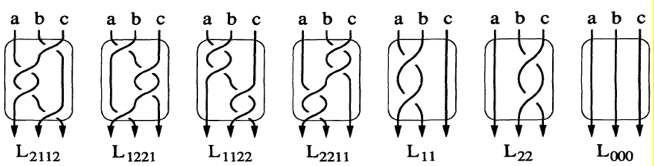
$\to \; xy(1-y+y^2)$

## Relations by J. Murakami (From [MJ])

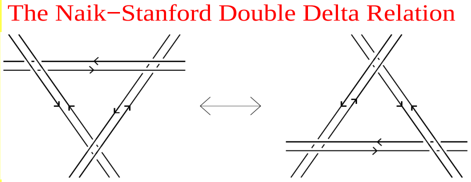


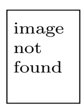




J. Murakami

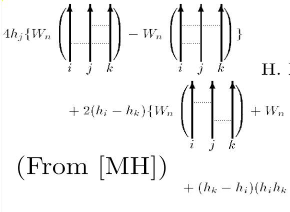



## The Naik−Stanford Double Delta Relation (From [NS])





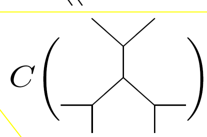

S. Naik T. Stanford

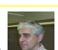


H. Murakami

(From [MH])

A Relation by H. Murakami



A Relation by A. Vaintrob (From [Va])

There's Lots More!

"God created the knots, all else in topology is the work of mortals"

Leopold Kronecker (paraphrased)


Visit! Edit!

http://katlas.org

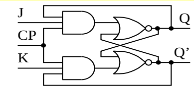
This handout and further links are at http://www.math.toronto.edu/~drorbn/Talks/Sandbjerg−0810/

**Our Goal.** Prove all these relations uniformly, at maximal confidence and minimal brain utilization.

$\Rightarrow$ We need an "Alexander Invariant" for arbitrary tangles, easy to define and compute and well-behaved under tangle compositions; better, "virtual tangles".

**Circuit Algebras**

* Have "circuits" with "ends",
* Can be wired arbitrarily.
* May have "relations" − de−Morgan, etc.




A J−K Flip Flop

Example $\quad V\mathcal{T}=CA\langle\times,\times\rangle/R23=PA\langle\times,\times,\times\rangle/R23,VR123,MR3$

**Reminders** from linear algebra. If $X$ is a (finite) set,

$$\Lambda^k(X):=\langle k\text{-tuples in } X,\text{ modulo anti-symmetry}\rangle$$

$$\Lambda^{\mathrm{top}}(X):=\langle |X|\text{-tuples in } X,\text{ modulo anti-symmetry}\rangle$$

$$\Lambda^{1/2}(X):=\langle (|X|/2)\text{-tuples in } X,\text{ modulo anti-symmetry}\rangle.$$

If $Y\subset X^m$, the "interior multiplication" $i_Y:\Lambda^k(X)\to\Lambda^{k-m}(X)$ is anti-symmetric in $Y$.

**Definition.** An "Alexander half density with input strands $X^{\mathrm{in}}$ and output strands $X^{\mathrm{out}}$" is an element of

$$\mathrm{AHD}(X^{\mathrm{in}},X^{\mathrm{out}}):=\Lambda^{\mathrm{top}}(X^{\mathrm{out}})\otimes\Lambda^{1/2}(X^{\mathrm{in}}\cup X^{\mathrm{out}}).$$

Often we extend the coefficients to some polynomial ring without warning.

**Definition.** If $\alpha_i\otimes p_i\in\mathrm{AHD}(X_i^{\mathrm{in}},X_i^{\mathrm{out}})$ (for $i=1,2$), and $G=(X_1^{\mathrm{in}}\cup X_2^{\mathrm{in}})\cap(X_1^{\mathrm{out}}\cup X_2^{\mathrm{out}})$ is the set of "gluable legs", the "gluing" in $\mathrm{AHD}(X_1^{\mathrm{in}}\cup X_2^{\mathrm{in}}-G,X_1^{\mathrm{out}}\cup X_2^{\mathrm{out}}-G)$ is

$$i_G(\alpha_1\wedge\alpha_2)\otimes i_G(p_1\wedge p_2).$$

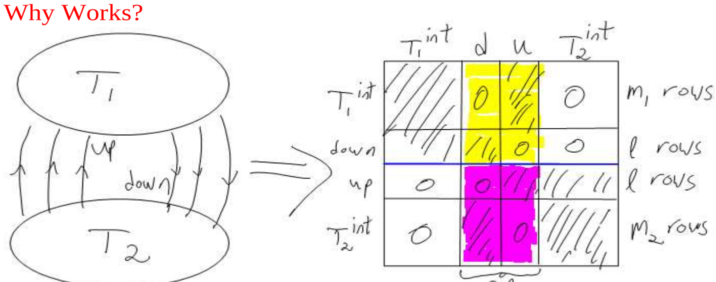
**Claim.** This makes AHD a circuit algebra.

**Definition.** The "Penultimate Alexander Invariant" is defined using

$$pA:\overset{k\ \ j}{\underset{l\ \ i}{\times}}\mapsto(j\wedge k)\otimes\begin{pmatrix}l\wedge i+(t_i-1)l\wedge j-t_l l\wedge k\\+i\wedge j+t_l j\wedge k\end{pmatrix}$$

$$pA:\overset{l\ \ k}{\underset{i\ \ j}{\times}}\mapsto(k\wedge l)\otimes\begin{pmatrix}t_j i\wedge j-t_j i\wedge l+j\wedge k\\+(t_i-1)j\wedge l+k\wedge l\end{pmatrix}$$

**Why Works?**




Every "rook arrangement" in the above picture must have exactly $l$ rooks in the yellow zone and $l$ rooks in the purple zone. So for $T_1$ we only care about the minors in which exactly $l$ of the $2l$ middle columns are dropped, and the rest is signs...

**Weaknesses.** Exponential, no understanding of cablings, no obvious "meaning". The ultimate Alexander invariant should address all that...

**Challenge.** Can you categorify this?