

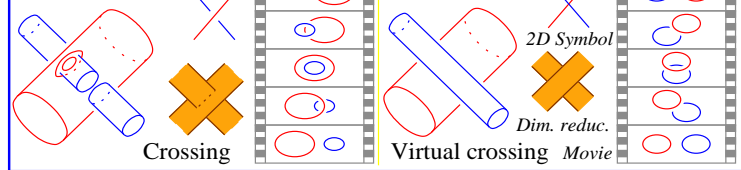
# w-Knots from Z to A

Dror Bar-Natan, Luminy, April 2010

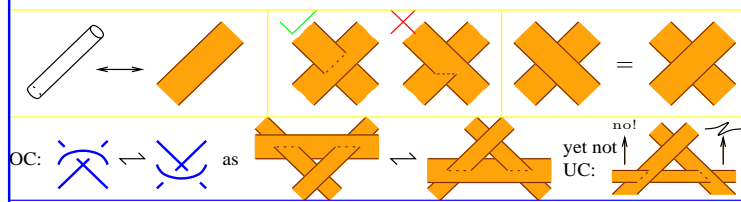
<http://www.math.toronto.edu/~drorbn/Talks/Luminy-1004/>

**Abstract** I will define w-knots, a class of knots wider than ordinary knots but weaker than virtual knots, and show that it is quite easy to construct a universal finite invariant of w-knots. In order to study Z we will introduce the “Euler Operator” and the “Infinitesimal Alexander Module”, at the end finding a simple determinant formula for Z. With no doubt that formula computes the Alexander polynomial A, except I don't have a proof yet.

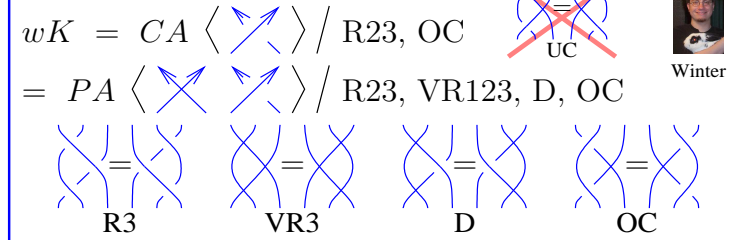
## Tubes in 4D.



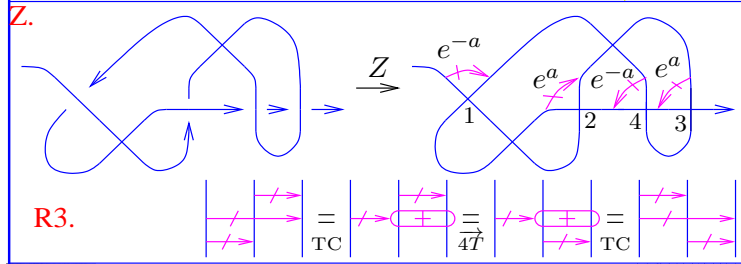
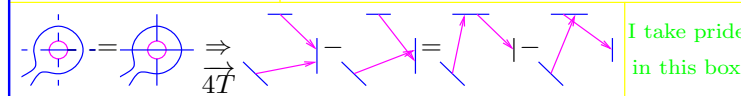
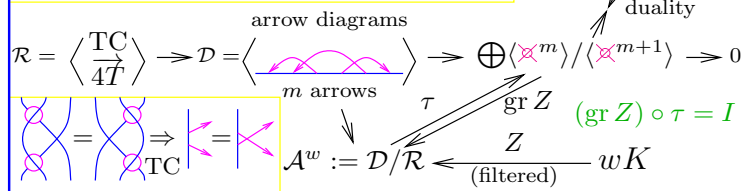
A Ribbon 2-Knot is a surface  $S$  embedded in  $\mathbb{R}^4$  that bounds an immersed handlebody  $B$ , with only “ribbon singularities”; a ribbon singularity is a disk  $D$  of transverse double points, whose preimages in  $B$  are a disk  $D_1$  in the interior of  $B$  and a disk  $D_2$  with  $D_2 \cap \partial B = \partial D_2$ , modulo isotopies of  $S$  alone.



## w-Knots.



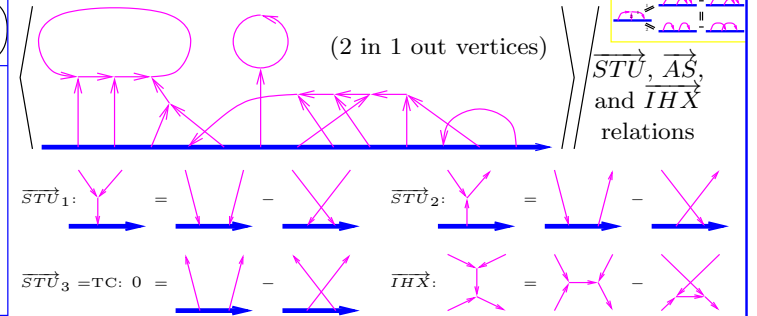
**The Finite Type Story.** With  $\bowtie := \times - \times$  set  $\mathcal{V}_m := \{V : wK \rightarrow \mathbb{Q} : V(\bowtie > m) = 0\}$ .



“God created the knots, all else in topology is the work of mortals.” Leopold Kronecker (modified)

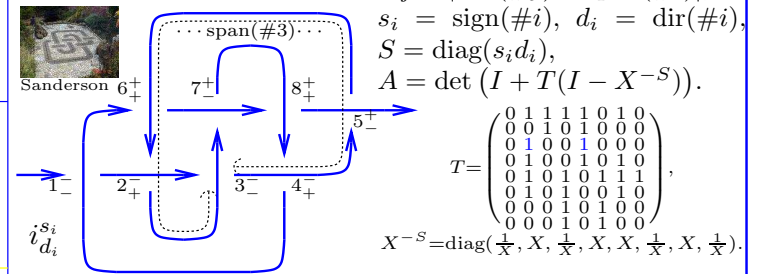
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## The Bracket-Rise Theorem. $\mathcal{A}^w$ is isomorphic to



**Corollaries.** (1) Related to Lie algebras! (2) Only wheels and isolated arrows persist. Habiro - can you do better?

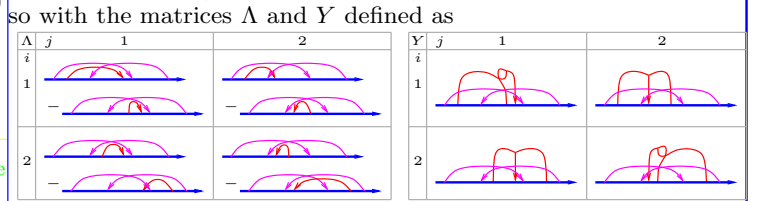
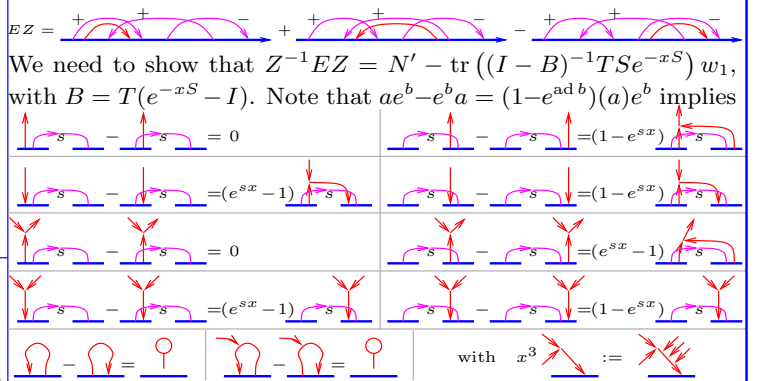
## The Alexander Theorem.



**Conjecture.** For u-knots,  $A$  is the Alexander polynomial.

**Theorem.** With  $w : x^k \mapsto w_k$  (the  $k$ -wheel),  $Z = N \exp_{\mathcal{A}^w}(-w(\log_{\mathbb{Q}[[x]]} A(e^x)))$  mod  $w_k w_l = w_{k+l}$ ,  $Z = N \cdot A^{-1}(e^x)$

**Proof Sketch.** Let  $E$  be the Euler operator, “multiply anything by its degree”,  $f \mapsto x f'$  in  $\mathbb{Q}[[x]]$ , so  $E e^x = x e^x$  and



we have  $EZ - N'' = \text{tr}(S\Lambda)$ ,  $\Lambda = -BY - Te^{-xS}w_1$ , and  $Y = BY + Te^{-xS}w_1$ . The theorem follows.

**So What?** • Habiro-Shima did this already, but not quite. (HS: *Finite Type Invariants of Ribbon 2-Knots, II*, Top. and its Appl. **111** (2001).)  
 • New (?) formula for Alexander, new (?) “Infinitesimal Alexander Module”. Related to Lescop’s arXiv:1001.4474?  
 • An “ultimate Alexander invariant”: local, composes well, behaves under cabling. Ought to also generalize the multi-variable Alexander polynomial and the theory of Milnor linking numbers.  
 • Tip of the Alekseev-Torossian-Kashiwara-Vergne iceberg (AT: *The Kashiwara-Vergne conjecture and Drinfeld’s associators*, arXiv:0802.4300).  
 • Tip of the v-knots iceberg. May lead to other polynomial-time polynomial invariants. “A polynomial’s worth a thousand exponentials”. Also see <http://www.math.toronto.edu/~drorbn/papers/WKO/>