Day 2 – u, v, w: combinatorics, low and high algebra

Dror Bar–Natan, Goettingen, April 2010
http://www.math.toronto.edu/~drorbn/Talks/Goettingen–1004/
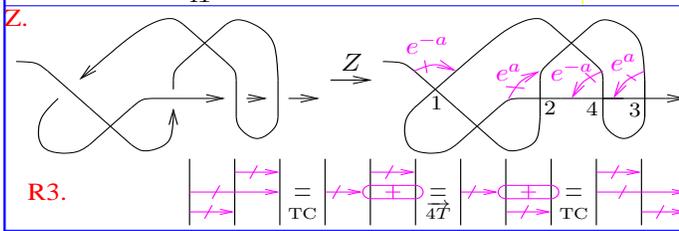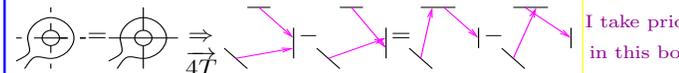
The Scheme. Topology → Combinatorics → Lie Theory via

$$\mathcal{K} \xrightarrow[\text{equations, unknowns}]{Z:\ \text{high algebra}} \mathcal{A} = \text{proj}\,\mathcal{K} = \bigoplus \mathcal{I}^m/\mathcal{I}^{m+1} \xrightarrow[\text{pictures} \to \text{formulas}]{\mathcal{T}_{\mathfrak{g}}:\ \text{low algebra}} \text{"}\mathcal{U}(\mathfrak{g})\text{"}$$

$1+1=2$, on an abacus, implies Duflo's $\mathcal{U}(\mathfrak{g})^{\mathfrak{g}} \cong \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ (with T. Le and D. Thurston).
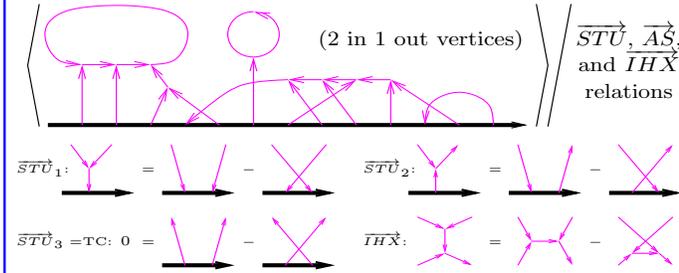

The Finite Type Story. With $\mathbb{X} := \times - \times$ set $\mathcal{V}_m := \{V : wK \to \mathbb{Q} : V(\mathbb{X}^{>m}) = 0\}$. $\bigoplus (\mathcal{V}_m/\mathcal{V}_{m-1})^*$

arrow diagrams

$$\mathcal{R} = \left\langle \frac{TC}{\overrightarrow{4T}} \right\rangle \longrightarrow \mathcal{D} = \left\langle \underbrace{}_{m\ \text{arrows}} \right\rangle \longrightarrow \bigoplus \langle \mathbb{X}^m \rangle / \langle \mathbb{X}^{m+1} \rangle \longrightarrow 0$$

$$\xrightarrow{\pi} \qquad \xrightarrow{\text{gr}\, Z} \qquad (\text{gr}\, Z) \circ \pi = I$$

$$\mathcal{A}^w := \mathcal{D}/\mathcal{R} \xleftarrow[\text{(filtered)}]{Z} wK$$

I take pride in this box

Z.

R3.

The Bracket-Rise Theorem. $\mathcal{A}^w$ is isomorphic to

(2 in 1 out vertices) $\overrightarrow{STU}, \overrightarrow{AS},$ and $\overrightarrow{IHX}$ relations

Corollaries. (1) Related to Lie algebras! (2) Only wheels and isolated arrows persist.
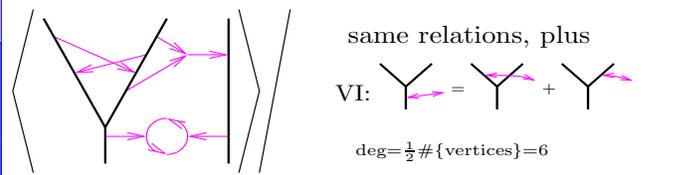
Low Algebra. With (x_i) and (φ^j) dual bases of $\mathfrak{g}$ and $\mathfrak{g}^*$ and with $[x_i, x_j] = \sum b_{ij}^k x_k$, we have $\mathcal{A}^w \to \mathcal{U}$ via

Penrose   Cvitanovic

$$\longrightarrow \sum_{i,j,k,l,m,n=1}^{\dim \mathfrak{g}} b_{ij}^k b_{kl}^m \varphi^i \varphi^j x_n x_m \varphi^l \in \mathcal{U}(I\mathfrak{g} := \mathfrak{g}^* \rtimes \mathfrak{g})$$

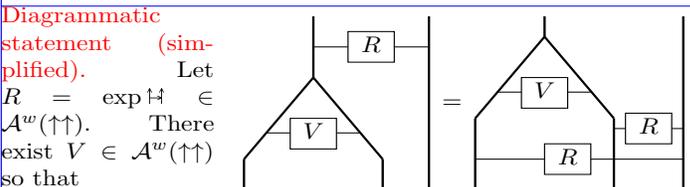**Vertices**   **R4**
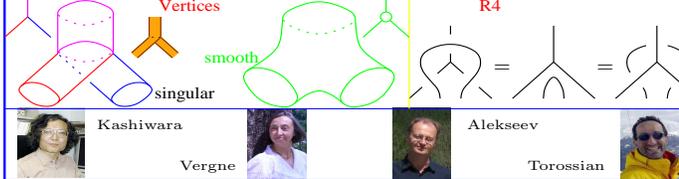
Kashiwara   Vergne   Alekseev   Torossian

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w-Jacobi diagrams and $\mathcal{A}$. $\mathcal{A}^w(Y \uparrow) \cong \mathcal{A}^w(\uparrow\uparrow\uparrow)$ is

same relations, plus

VI:

$\deg = \frac{1}{2}\#\{\text{vertices}\} = 6$

Knot-Theoretic statement (simplified). There exists a homomorphic expansion Z for trivalent w-tangles. In particular, Z should respect $R4$.

Diagrammatic statement (simplified). Let $R = \exp \mapsfrom \in \mathcal{A}^w(\uparrow\uparrow)$. There exist $V \in \mathcal{A}^w(\uparrow\uparrow)$ so that

Algebraic statement (simplified). With $r \in \mathfrak{g}^* \otimes \mathfrak{g}$ the identity element and with $R = e^r \in \hat{\mathcal{U}}(I\mathfrak{g}) \otimes \hat{\mathcal{U}}(\mathfrak{g})$ there exist $V \in \hat{\mathcal{U}}(I\mathfrak{g})^{\otimes 2}$ so that $V(\Delta \otimes 1)(R) = R^{13}R^{23}V$ in $\hat{\mathcal{U}}(I\mathfrak{g})^{\otimes 2} \otimes \hat{\mathcal{U}}(\mathfrak{g})$

Unitary statement (simplified). There exists a unitary tangential differential operator V defined on $\text{Fun}(\mathfrak{g}_x \times \mathfrak{g}_y)$ so that $V\widehat{e^{x+y}} = \widehat{e^x}\widehat{e^y}V$ (allowing $\hat{\mathcal{U}}(\mathfrak{g})$-valued functions)

Unitary $\Longleftrightarrow$ Algebraic. Interpret $\hat{\mathcal{U}}(I\mathfrak{g})$ as tangential differential operators on $\text{Fun}(\mathfrak{g})$: $\varphi \in \mathfrak{g}^*$ becomes a multiplication operator, and $x \in \mathfrak{g}$ becomes a tangential derivation, in the direction of the action of $\text{ad}\, x$: $(x\varphi)(y) := \varphi([x, y])$.

Group-Algebra statement (simplified). For every $\phi, \psi \in \text{Fun}(\mathfrak{g})^G$ (with small support), the following holds in $\mathcal{U}(\mathfrak{g})$:

$$\iint_{\mathfrak{g} \times \mathfrak{g}} \phi(x)\psi(y)e^{x+y} = \iint_{\mathfrak{g} \times \mathfrak{g}} \phi(x)\psi(y)e^x e^y.$$

(shhh, this is Duflo)

Unitary $\Longrightarrow$ Group-Algebra. $\iint e^{x+y}\phi(x)\psi(y) = \langle 1, e^{x+y}\phi(x)\psi(y)\rangle = \langle V1, Ve^{x+y}\phi(x)\psi(y)\rangle = \langle 1, e^x e^y V\phi(x)\psi(y)\rangle = \langle 1, e^x e^y \phi(x)\psi(y)\rangle = \iint e^x e^y \phi(x)\psi(y).$

Convolutions statement (Kashiwara-Vergne, simplified). Convolutions of invariant functions on a Lie group agree with convolutions of invariant functions on its Lie algebra. More accurately, let G be a finite dimensional Lie group and let $\mathfrak{g}$ be its Lie algebra, and let $\Phi : \text{Fun}(G) \to \text{Fun}(\mathfrak{g})$ be given by $\Phi(f)(x) := f(\exp x)$. Then if $f, g \in \text{Fun}(G)$ are Ad-invariant and supported near the identity, then $\Phi(f) \star \Phi(g) = \Phi(f \star g)$.

Convolutions and Group Algebras (ignoring all Jacobians). If G is finite, A is an algebra, $\tau : G \to A$ is multiplicative then $(\text{Fun}(G), \star) \to (A, \cdot)$ via $L : f \mapsto \sum f(a)\tau(a)$. For Lie $(G, \mathfrak{g})$,

with $L_0\psi = \int \psi(x)e^x dx \in \hat{\mathcal{S}}(\mathfrak{g})$ and $L_1\Phi^{-1}\psi = \int \psi(x)e^x \in \hat{\mathcal{U}}(\mathfrak{g})$. Given $\psi_i \in \text{Fun}(\mathfrak{g})$ compare $\Phi^{-1}(\psi_1) \star \Phi^{-1}(\psi_2)$ and $\Phi^{-1}(\psi_1 \star \psi_2)$ in $\hat{\mathcal{U}}(\mathfrak{g})$: (shhh, $L_{0/1}$ are "Laplace transforms")

$\star$ in G: $\iint \psi_1(x)\psi_2(y)e^x e^y$ $\star$ in $\mathfrak{g}$: $\iint \psi_1(x)\psi_2(y)e^{x+y}$

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Goettingen-1004/