

**Definition.** A knot invariant is any function whose domain is {knots}. Really, we mean a computable function whose target space is understandable; e.g.

$$C: \left\{ \begin{array}{l} \text{Knots} \\ \text{with } \chi_1 = \chi_2 \end{array} \right\} \rightarrow \mathbb{Z}[z]$$

**Example.** The Conway polynomial is given by

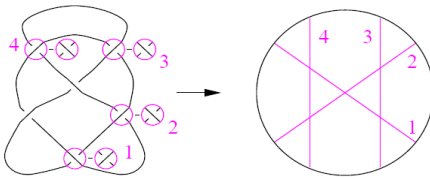
$$C(\text{crossing}) - C(\text{smoothing}) = z C(\text{other crossing})$$

$$\text{and } C(\text{link with } k \text{ crossings}) = \begin{cases} 1 & k=1 \\ 0 & k>1 \end{cases}$$

**Exercise.** Pick your favourite bank and compute the Conway polynomial of its logo.



**Definition.** Any  $V: \{\text{knots}\} \rightarrow \text{Abelian Group } A$  can be extended to "knots w/ double points" using  $V(\text{crossing}) = V(\text{smoothing}) - V(\text{other smoothing})$ . (Think "differentiation")



**Definition.**  $V$  is of type  $m$  if always  $V(\text{link with } m+1 \text{ crossings}) = 0$  (think "polynomial")

$$V(\text{link with } m+1 \text{ crossings}) = 0$$

**Conjecture.** Finite type invariants separate knots.

**Theorem.** If  $C(K) = \sum_{m=0}^{\infty} V_m(K) z^m$  then  $V_m$  is of type  $m$ .

**Proof.**  $C(\text{crossing}) = C(\text{smoothing}) - C(\text{other smoothing}) = z C(\text{other crossing}) \square$

Let  $V$  be of type  $m$ ; then  $V^{(m)}$  is constant:

$$V(\text{link with } m \text{ crossings}) = V(\text{link with } m-1 \text{ crossings})$$

So  $W_V := V^{(m)} = V|_{\text{m-singular knots}}$  is really a function on  $m$ -chord diagrams:  $W_V: \{\text{m-chord diagrams}\} \rightarrow A$

**Claim.**  $W_V$  satisfies the 4T relation:

$W_V(\text{diagram 1}) - W_V(\text{diagram 2}) - W_V(\text{diagram 3}) + W_V(\text{diagram 4}) = 0$

$$W_V(\text{diagram 1}) - W_V(\text{diagram 2}) - W_V(\text{diagram 3}) + W_V(\text{diagram 4}) = 0$$

**Proof.**  $V(\text{link with } m-2 \text{ crossings}) = V(\text{link with } m-2 \text{ crossings}) \square$

**Exercise for Lecture 2.** Use  $\int_{\mathbb{R}^n} e^{-x^2/2} = \sqrt{2\pi}$ , Fubini's theorem, and polar coordinates to compute  $\int_{\mathbb{R}^n} e^{-\|x\|^2/2} dx$  in two different ways and hence to deduce the volume of  $S^{n-1}$ , the  $(n-1)$ -dimensional sphere.

**Exercise.** 1. Determine the "weight system"  $W_m$  of the  $m$ -th coefficient of the Conway polynomial and verify that it satisfies 4T. 2. Learn somewhere about the Jones polynomial, and do the same for its coefficients.

**Theorem. (The Fundamental Theorem)**

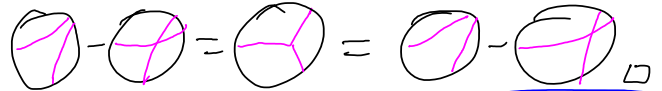
Every "weight system", i.e. every linear functional  $W$  on  $\mathcal{A} := \{\text{chord diagrams}\} / 4T$  is the  $m$ th derivative of a type  $m$  invariant:  $\forall W \exists V$  s.t.  $W = W_V$



$m$	0	1	2	3	4	5	6	7	8	9	10	11	12
$\dim \mathcal{A}_m^r$	1	0	1	1	3	4	9	14	27	44	80	132	232
$\dim \mathcal{A}_m$	1	1	2	3	6	10	19	33	60	104	184	316	548
$\dim \mathcal{P}_m$	0	1	1	1	2	3	5	8	12	18	27	39	55

**Theorem.**  $\mathcal{A}^{\text{today}} \cong \mathcal{A}^{\text{Monday}}$

**Proof**



**Proposition.** The fundamental theorem holds iff there exists an expansion:  $Z: \mathcal{K} \rightarrow \hat{\mathcal{A}}$  s.t. if  $K$  is  $m$ -singular, then  $Z(K) = D_K + \text{higher degrees}$ .

**Proof.**  $\mathcal{K} \xrightarrow{Z} \hat{\mathcal{A}}$   
 $\downarrow W$   
 $\mathcal{Q} \square$

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KnotTheory
Loading KnotTheory' version of August 22, 2010, 13:36:57.55.
Read more at http://katlas.org/wiki/KnotTheory.

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Also see my old paper, "On the Vassiliev knot invariants" (google will find...)