

**Table 18-1 Classical Physics**

# A Bit on Maxwell's Equations

## Prerequisites.

- Poincaré's Lemma, which says that on  $\mathbb{R}^n$ , every closed form is exact. That is, if  $d\omega = 0$ , then there exists  $\eta$  with  $d\eta = \omega$ .
- Integration by parts:  $\int \omega \wedge d\eta = -(-1)^{\deg \omega} \int (d\omega) \wedge \eta$  on domains that have no boundary.
- The Hodge star operator  $\star$  which satisfies  $\omega \wedge \star \eta = \langle \omega, \eta \rangle dx_1 \cdots dx_n$  whenever  $\omega$  and  $\eta$  are of the same degree.
- The simplest least action principle: the extremes of  $q \mapsto \int_a^b (\frac{1}{2}m\dot{q}^2(t) - V(q(t))) dt$  occur when  $m\ddot{q} = -V'(q(t))$ . That is, when  $F = ma$ .

<b>Maxwell's equations</b>	
I. $\nabla \cdot E = \frac{\rho}{\epsilon_0}$	(Flux of $E$ through a closed surface) = (Charge inside)/ $\epsilon_0$
II. $\nabla \times E = -\frac{\partial B}{\partial t}$	(Line integral of $E$ around a loop) = $-\frac{d}{dt}$ (Flux of $B$ through the loop)
III. $\nabla \cdot B = 0$	(Flux of $B$ through a closed surface) = 0
IV. $c^2 \nabla \times B = \frac{J}{\epsilon_0} + \frac{\partial E}{\partial t}$	$c^2$ (Integral of $B$ around a loop) = (Current through the loop)/ $\epsilon_0$ + $\frac{\partial}{\partial t}$ (Flux of $E$ through the loop)
[Conservation of charge $\nabla \cdot j = -\frac{\partial \rho}{\partial t}$ (Flux of current through a closed surface) = $-\frac{\partial}{\partial t}$ (Charge inside)]	
<b>Force law</b> $F = q(E + v \times B)$	
<b>Law of motion</b> $\frac{d}{dt}(p) = F$ , where $p = \frac{mv}{\sqrt{1 - v^2/c^2}}$ (Newton's law, with Einstein's modification)	
<b>Gravitation</b> $F = -G \frac{m_1 m_2}{r^2} e_r$	

The Feynman Lectures on Physics vol. II, page 18-2

**The Action Principle.** The *Vector Field* is a compactly supported 1-form  $A$  on  $\mathbb{R}^4$  which extremizes the *action*

$$S_J(A) := \int_{\mathbb{R}^4} \frac{1}{2} \|dA\|^2 dt dx dy dz + J \wedge A$$

where the 3-form  $J$  is the *charge-current*.

**The Euler-Lagrange Equations** in this case are  $d \star dA = J$ , meaning that there's no hope for a solution unless  $dJ = 0$ , and that we might as well (think Poincaré's Lemma!) change variables to  $F := dA$ . We thus get

$$dJ = 0 \quad dF = 0 \quad d \star F = J$$

**These are the Maxwell equations!** Indeed, writing  $F = (E_x dxdt + E_y dydt + E_z dzdt) + (B_x dydz + B_y dzdx + B_z dx dy)$  and  $J = \rho dx dy dz - j_x dy dz dt - j_y dz dx dt - j_z dx dy dt$ , we find:

$dJ = 0 \implies$	$\frac{\partial \rho}{\partial t} + \text{div } j = 0$	"conservation of charge"
$dF = 0 \implies$	$\text{div } B = 0$	"no magnetic monopoles"
	$\text{curl } E = -\frac{\partial B}{\partial t}$	that's how generators work!
$d \star F = J \implies$	$\text{div } E = -\rho$	"electrostatics"
	$\text{curl } B = -\frac{\partial E}{\partial t} + j$	that's how electromagnets work!

**Exercise.** Use the Lorentz metric to fix the sign errors.

**Exercise.** Use pullbacks along Lorentz transformations to figure out how  $E$  and  $B$  (and  $j$  and  $\rho$ ) appear to moving observers.

**Exercise.** With  $ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$  use  $S = mc \int_{e_1}^{e_2} (ds + eA)$  to derive Feynman's "law of motion" and "force law".