

## Gaussian Integration, Determinants, Feynman Diagrams

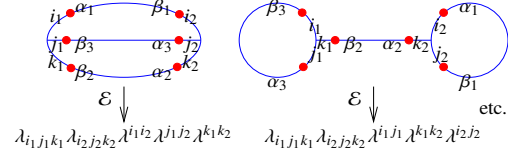
**Gaussian Integration.**  $(\lambda_{ij})$  is a symmetric positive definite matrix and  $(\lambda^{ij})$  is its inverse, and  $(\lambda_{ijk})$  are the coefficients of some cubic form. Denote by  $(x^i)_{i=1}^n$  the coordinates of  $\mathbb{R}^n$ , let  $(t_i)_{i=1}^n$  be a set of “dual” variables, and let  $\partial^i$  denote  $\frac{\partial}{\partial t_i}$ . Also let  $C := \frac{(2\pi)^{n/2}}{\det(\lambda_{ij})}$ . Then

$$\int_{\mathbb{R}^n} e^{-\frac{1}{2}\lambda_{ij}x^i x^j + \frac{\epsilon}{6}\lambda_{ijk}x^i x^j x^k} = \sum_{m \geq 0} \frac{\epsilon^m}{6^m m!} \int_{\mathbb{R}^n} (\lambda_{ijk}x^i x^j x^k)^m e^{-\frac{1}{2}\lambda_{ij}x^i x^j}$$

$$= \sum_{m \geq 0} \frac{C \epsilon^m}{6^m m!} (\lambda_{ijk} \partial^i \partial^j \partial^k)^m e^{\frac{1}{2}\lambda^{\alpha\beta} t_\alpha t_\beta} \Big|_{t_\alpha=0} = \sum_{\substack{m, l \geq 0 \\ 3m=2l}} \frac{C \epsilon^m}{6^m m! 2^l l!} (\lambda_{ijk} \partial^i \partial^j \partial^k)^m (\lambda^{\alpha\beta} t_\alpha t_\beta)^l$$

Feynman 

$$= \sum_{\substack{m, l \geq 0 \\ 3m=2l}} \frac{C \epsilon^m}{6^m m! 2^l l!} \left[ \begin{array}{c} \lambda^{\alpha_1 \beta_1} \quad \lambda^{\alpha_2 \beta_2} \quad \lambda^{\alpha_3 \beta_3} \quad \dots \quad \lambda^{\alpha_l \beta_l} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \\ t_{\alpha_1} \quad t_{\beta_1} \quad t_{\alpha_2} \quad t_{\beta_2} \quad t_{\alpha_3} \quad t_{\beta_3} \quad \dots \quad t_{\alpha_l} \quad t_{\beta_l} \\ \dots \text{ sum over all pairings } \dots \\ \partial^{j_1} \quad \partial^{j_2} \quad \partial^{j_3} \quad \dots \quad \partial^{j_m} \quad \partial^{j_{m+1}} \quad \partial^{j_{m+2}} \quad \dots \quad \partial^{j_{2m}} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \\ \lambda_{i_1 j_1 k_1} \quad \lambda_{i_2 j_2 k_2} \quad \dots \quad \lambda_{i_m j_m k_m} \end{array} \right]$$

$$= \sum_{\substack{m, l \geq 0 \\ 3m=2l}} \frac{C \epsilon^m}{6^m m! 2^l l!} \sum_{\substack{m\text{-vertex fully marked} \\ \text{Feynman diagrams } D}} \mathcal{E}(D)$$


$$= C \sum_{\text{unmarked Feynman diagrams } D} \frac{\epsilon^{m(D)} \mathcal{E}(D)}{|\text{Aut}(D)|}$$

**Claim.** The number of pairings that produce a given unmarked Feynman diagram  $D$  is  $\frac{6^m m! 2^l l!}{|\text{Aut}(D)|}$ .

**Proof of the Claim.** The group  $G_{m,l} := [(S_3)^m \times S_m] \times [(S_2)^l \times S_l]$  acts on the set of pairings, the action is transitive on the set of pairings  $P$  that produce a given  $D$ , and the stabilizer of any given  $P$  is  $\text{Aut}(D)$ .  $\square$

**Determinants.** Now suppose  $Q$  and  $P_i$  ( $1 \leq i \leq n$ ) are  $d \times d$  matrices and  $Q$  is invertible. Then

$$|Q|^{-1} I_{\epsilon, \lambda_{ij}, \lambda_{ijk}, Q, P_i} = |Q|^{-1} \int_{\mathbb{R}^n} e^{-\frac{1}{2}\lambda_{ij}x^i x^j + \frac{\epsilon}{6}\lambda_{ijk}x^i x^j x^k} \det(Q + \epsilon x^i P_i)$$

$$= \sum_{m, k \geq 0, \sigma \in S_k} \frac{C \epsilon^{m+k} (-)^\sigma}{6^m m! k!} \int_{\mathbb{R}^n} (\lambda_{ijk} x^i x^j x^k)^m \text{tr}(\sigma(x^i Q^{-1} P_i)^{\otimes k}) e^{-\frac{1}{2}\lambda_{ij}x^i x^j}$$

$$= \sum_{\text{fully marked Feynman diagrams}} \frac{C \epsilon^{m+k} (-)^\sigma}{6^m m! k!} \mathcal{E} \left( \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \\ \sigma \in S_k \\ \text{---} \text{---} \text{---} \text{---} \text{---} \\ Q^{-1} \quad Q^{-1} \quad Q^{-1} \quad Q^{-1} \end{array} \right)$$

$$= \sum_{\text{Feynman diagrams}} C \epsilon^{m+k} (-)^k (-)^l \mathcal{E} \left( \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \end{array} \right),$$

where  $l$  is the number of purple (“Fermion”) loops.

**Ghosts.** Or else, introduce “ghosts”  $\bar{c}_a$  and  $c^b$ , write

$$I_{\epsilon, \lambda_{ij}, \lambda_{ijk}, Q, P_i} = \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{-\frac{1}{2}\lambda_{ij}x^i x^j + \frac{\epsilon}{6}\lambda_{ijk}x^i x^j x^k + \bar{c}_a(Q_a^b + \epsilon x^i P_{ib}^a) c^b}$$

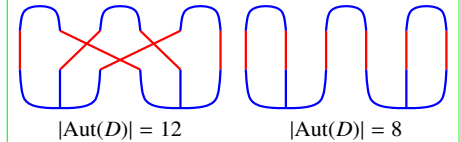


and use “ordinary” perturbation theory.

## The Fourier Transform.

$(F: V \rightarrow \mathbb{C}) \Rightarrow (\tilde{f}: V^* \rightarrow \mathbb{C})$   
 via  $\tilde{F}(\varphi) := \int_V f(v) e^{-i\langle \varphi, v \rangle} dv$ . Some facts:  
 •  $\tilde{f}(0) = \int_V f(v) dv$ .  
 •  $\frac{\partial}{\partial \varphi_i} \tilde{f} \sim \tilde{v}^i f$ .  
 •  $(e^{Q/2}) \sim e^{Q^{-1}/2}$ , where  $Q$  is quadratic,  $Q(v) = \langle Lv, v \rangle$  for  $L: V \rightarrow V^*$ , and  $Q^{-1}(\varphi) := \langle \varphi, L^{-1}\varphi \rangle$ . (This is the key point in the proof of the Fourier inversion formula!)

### Examples.

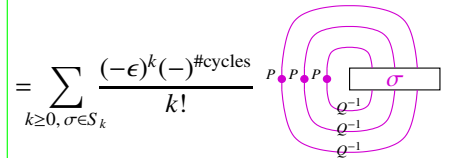


**Perturbing Determinants.** If  $Q$  and  $P$  are matrices and  $Q$  is invertible,

$$|Q|^{-1} |Q + \epsilon P| = |I + \epsilon Q^{-1} P|$$

$$= \sum_{k \geq 0} \epsilon^k \text{tr} \left( \bigwedge^k Q^{-1} P \right)$$

$$= \sum_{k \geq 0, \sigma \in S_k} \frac{\epsilon^k (-)^\sigma}{k!} \text{tr}(\sigma(Q^{-1} P)^{\otimes k})$$



$$= \sum_{k \geq 0, \sigma \in S_k} \frac{(-\epsilon)^k (-)^{\#\text{cycles}}}{k!} P_{\sigma} \dots P_{\sigma} \begin{array}{c} \sigma \\ \text{---} \\ Q^{-1} \\ \text{---} \\ Q^{-1} \\ \text{---} \\ Q^{-1} \end{array}$$

**The Berezin Integral** (physics / math language, formulas from Wikipedia: Grassmann integral).

**The Berezin Integral** is linear on functions of anti-commuting variables, and satisfies  $\int d\theta = 1$ , and  $\int 1 d\theta = 0$ , so that  $\int \frac{\partial f(\theta)}{\partial \theta} d\theta = 0$ .

Let  $V$  be a vector space,  $\theta \in V$ ,  $d\theta \in V^*$  s.t.  $\langle d\theta, \theta \rangle = 1$ . Then  $f \mapsto \int f d\theta$  is the interior multiplication map  $\wedge V \rightarrow \wedge V: \int f d\theta := i_{d\theta}(f)$  ( $= \frac{\partial f}{\partial \theta}$ ).

Multiple integration via “Fubini”:  $\int f_1(\theta_1) \dots f_n(\theta_n) d\theta_1 \dots d\theta_n := (\int f_1 d\theta_1) \dots (\int f_n d\theta_n)$ .  $\int f d\theta_1 \dots d\theta_n := f \parallel i_{d\theta_1} \parallel \dots \parallel i_{d\theta_n}$ .

Change of variables. If  $\theta_i = \theta_i(\xi_j)$ , both  $\theta_i$  and  $\xi_j$  are odd, and  $J_{ij} := \partial \theta_i / \partial \xi_j$ , then

$$\int f(\theta_i) d\theta = \int f(\theta_i(\xi_j)) \det(J_{ij})^{-1} d\xi.$$

Given vector spaces  $V_{\theta_i}$  and  $W_{\xi_j}$ ,  $d\theta = \wedge d\theta_i \in \wedge^{\text{top}}(V^*)$ ,  $d\xi = \wedge d\xi_j \in \wedge^{\text{top}}(W^*)$ , and  $T: V \rightarrow \wedge^{\text{odd}}(W)$ . Then  $T$  induces a map  $T_*: \wedge V \rightarrow \wedge W$  and then

$$\int f d\theta = \int (T_* f) \det \left( \frac{\partial (T \theta_i)}{\partial \xi_j} \right)^{-1} d\xi.$$

**Gaussian integration.** For an even matrix  $A$  and odd vectors  $\theta, \eta$ ,  $\int e^{\theta^T A \eta} d\theta d\eta = \det(A)$ ,  $\int e^{\theta^T A \eta + \theta^T J + K^T \eta} d\theta d\eta = \det(A) e^{-K^T A^{-1} J}$ .



Berezin