

**Simple 2-Knots.**

“broken surface diagram”  
 A 4D knot by Carter and Saito [CS]

$\omega\epsilon\beta/F$

Dalvit  
 $\omega\epsilon\beta/Dal$

**Question.** Does it all extend to arbitrary 2-knots (not necessarily “simple”)? To arbitrary codimension-2 knots?

**BF Following [CR].**  $A \in \Omega^1(M = \mathbb{R}^4, \mathfrak{g}), B \in \Omega^2(M, \mathfrak{g}^*),$

$$S(A, B) := \int_M \langle B, F_A \rangle.$$

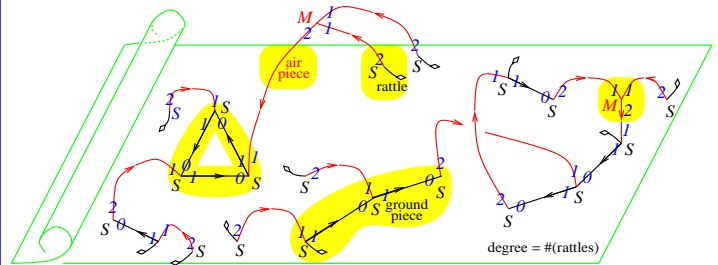
With  $\kappa: (S = \mathbb{R}^2) \rightarrow M, \beta \in \Omega^0(S, \mathfrak{g}), \alpha \in \Omega^1(S, \mathfrak{g}^*),$  set

$$O(A, B, \kappa) := \int \mathcal{D}\beta \mathcal{D}\alpha \exp\left(\frac{i}{\hbar} \int_S \langle \beta, d_{\kappa^* A} \alpha + \kappa^* B \rangle\right).$$

**The BF Feynman Rules.** For an edge  $e,$  let  $\Phi_e$  be its direction, in  $S^3$  or  $S^1.$  Let  $\omega_3$  and  $\omega_1$  be volume forms on  $S^3$  and  $S^1.$  Then

$$Z_{BF} = \sum_{\text{diagrams } D} \frac{|D|}{|\text{Aut}(D)|} \int_{\mathbb{R}^2} \dots \int_{\mathbb{R}^2} \int_{\mathbb{R}^4} \dots \int_{\mathbb{R}^4} \prod_{\text{red } e \in D} \Phi_e^* \omega_3 \prod_{\text{black } e \in D} \Phi_e^* \omega_1$$

(modulo some IHX-like relations). See also [Wa]



- Issues.**
- Signs don't quite work out, and BF seems to reproduce only “half” of the wheels invariant on simple 2-knots.
  - There are many more configuration space integrals than BF Feynman diagrams and than just trees and wheels.
  - I don't know how to define / analyze “finite type” for general 2-knots.
  - I don't know how to reduce  $Z_{BF}$  to combinatorics / algebra.

**References.**

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 [Le] J. Levine, *A Factorization of the Conway Polynomial,* Comment. Math. Helv. **74** (1999) 27–53, arXiv:q-alg/9711007.  
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**The Generators**

“the crossing”  $\omega\epsilon\beta/X$

“v-xing”  $\omega\epsilon\beta/vX$

“cap”  $\delta$

**The Double Inflation Procedure  $\delta.$**

**w-Knots.**

$w\mathcal{K} := \text{PA} \langle \dots \rangle$

Is this All???

OC:  $\dots$  as  $\dots$  yet not UC:  $\dots$

**A Big Open Problem.**  $\delta$  maps w-knots onto simple 2-knots. To what extent is it a bijection? What other relations are required? In other words, **find a simple description of simple 2-knots.**

**The Full 2-Knot Story**

**Rewrites of IHX.**

Riddles, in case you are bored.

- Can you find uncountably many distinct subsets  $\{A_\alpha\}$  of  $\mathbb{Z}$  such that whenever  $\alpha \neq \beta$  either  $A_\alpha \subset A_\beta$  or  $A_\beta \subset A_\alpha$ ?
- Can you find uncountably many distinct subsets  $\{B_\alpha\}$  of  $\mathbb{Z}$  such that whenever  $\alpha \neq \beta$  the intersection  $B_\alpha \cap B_\beta$  is finite?

Even better,

“God created the knots, all else in topology is the work of mortals.”  
 Leopold Kronecker (modified)

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