Dror Bar-Natan: Talks: LesDiablerets-1508:
ωεβ:=http://www.math.toronto.edu/~drorbn/Talks/LesDiablerets-1508/

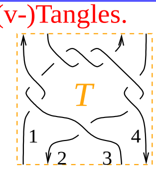
Work in Progress on

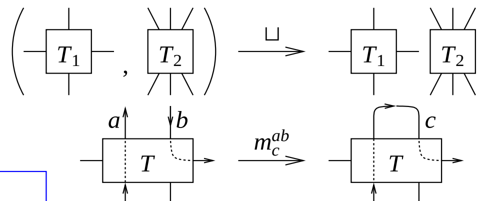
# Polynomial Time Knot Polynomials, A

**Abstract.** The value of things is inversely correlated with their computational complexity. "Real time" machines, such as our brains, only run linear time algorithms, and there's still a lot we don't know. Anything we learn about things doable in linear time is truly valuable. Polynomial time we can in-practice run, even if we have to wait; these things are still valuable. Exponential time we can play with, but just a little, and exponential things must be beautiful or philosophically compelling to deserve attention. Values further diminish and the aesthetic-or-philosophical bar further rises as we go further slower, or un-computable, or ZFC-style intrinsically infinite, or large-cardinalish, or beyond.

I will explain some things I know about polynomial time knot polynomials and explain where there's more, within reach.

## (v-)Tangles.




## Why Tangles?

- Finitely presented. (meta-associativity: $m_a^{ab} /\!/ m_a^{ac} = m_b^{bc} /\!/ m_a^{ab}$)
- Divide and conquer proofs and computations.
- "Algebraic Knot Theory": If $K$ is ribbon, $z(K) \in \{cl_2(\zeta) \colon cl_1(\zeta) = 1\}$.

(Genus and crossing number are also definable properties).

$U \in \mathcal{T}_n \xrightarrow{cl_1}$, $\mathcal{T}_{2n} \xrightarrow{cl_2} K \in \mathcal{T}_1$; $cl_1$: trivial, $cl_2$: ribbon

Faster is better, leaner is meaner!

A blackboard aside on genus?

## Theorem 1.

∃! an invariant $z_0$: {pure framed $S$-component tangles} $\to \Gamma_0(S) := R \times M_{S\times S}(R)$, where $R = R_S = \mathbb{Z}((T_a)_{a\in S})$ is the ring of rational functions in $S$ variables, intertwining

$$\left(\begin{array}{c|c}\omega_1 & S_1\\ \hline S_1 & A_1\end{array}, \begin{array}{c|c}\omega_2 & S_2\\ \hline S_2 & A_2\end{array}\right) \xrightarrow{\sqcup} \begin{array}{c|cc}\omega_1\omega_2 & S_1 & S_2\\ \hline S_1 & A_1 & 0\\ S_2 & 0 & A_2\end{array},$$

$$\begin{array}{c|ccc}\omega & a & b & S\\ \hline a & \alpha & \beta & \theta\\ b & \gamma & \delta & \epsilon\\ S & \phi & \psi & \Xi\end{array} \xrightarrow[T_a, T_b \to T_c,\ \mu := 1-\beta]{m_c^{ab}} \left(\begin{array}{c|cc}\mu\omega & c & S\\ \hline c & \gamma + \alpha\delta/\mu & \epsilon + \delta\theta/\mu\\ S & \phi + \alpha\psi/\mu & \Xi + \psi\theta/\mu\end{array}\right),$$

and satisfying $\left(|_a;\ {}_a\nearrow_b,\ {}_b\nwarrow_a\right) \xrightarrow{z_0} \left(\begin{array}{c|c}1 & a\\ \hline a & 1\end{array}; \begin{array}{c|cc}1 & a & b\\ \hline a & 1 & 1-T_a^{\pm 1}\\ b & 0 & T_a^{\pm 1}\end{array}\right).$

## In Addition

- The matrix part is just a stitching formula for Burau/Gassner [LD, KLW, CT].
- $K \mapsto \omega$ is Alexander, mod units.
- $L \mapsto (\omega, A) \mapsto \omega \det'(A - I)/(1 - T')$ is the MVA, mod units.
- The fastest Alexander algorithm I know.
- There are also formulas for strand deletion, reversal, and doubling.

M. Polyak & T. Ohtsuki @ Heian Shrine, Kyoto

- Every step along the computation is the invariant of something.
- Extends to and more naturally defined on v/w-tangles.
- Fits in one column, including propaganda & implementation.

## Implementation key idea:

ωεβ/Demo

$(\omega, A = (\alpha_{ab})) \leftrightarrow$
$(\omega, \lambda = \sum \alpha_{ab} t_a h_b)$

```
ΓCollect[Γ[ω_, λ_]] := Γ[Simplify[ω],
  Collect[λ, h_, Collect[#, t_, Factor] &]];
Format[Γ[ω_, λ_]] := Module[{S, M},
  S = Union@Cases[Γ[ω, λ], (h | t)_a_ :> a, ∞];
  M = Outer[Factor[∂_{h_{#1} t_{#2}} λ] &, S, S];
  M = Prepend[M, t_# & /@ S] // Transpose;
  M = Prepend[M, Prepend[h_# & /@ S, ω]];
  M // MatrixForm];
```

```
Γ /: Γ[ω1_, λ1_] Γ[ω2_, λ2_] := Γ[ω1*ω2, λ1+λ2];
m_{a_,b_→c_}[Γ[ω_, λ_]] := Module[{α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ},
  (α β θ; γ δ ϵ; ϕ ψ Ξ) = (∂_{t_a,h_a}λ ∂_{t_a,h_b}λ ∂_{t_a}λ; ∂_{t_b,h_a}λ ∂_{t_b,h_b}λ ∂_{t_b}λ; ∂_{h_a}λ ∂_{h_b}λ λ) /. (t | h)_{a|b} → 0;
  Γ[(μ = 1 - β) ω, {t_c, 1}.(γ + α δ / μ   ϵ + δ θ / μ; ϕ + α ψ / μ   Ξ + ψ θ / μ).{h_c, 1}]
    /. {T_a → T_c, T_b → T_c} // ΓCollect];
Rp_{a_,b_} := Γ[1, {t_a, t_b}.(1  1 - T_a; 0  T_a).{h_a, h_b}];
Rm_{a_,b_} := Rp_{ab} /. T_a → 1 / T_a;
```

## Meta-Associativity

**Runs.**

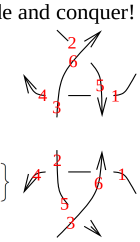
```
ζ = Γ[ω, {t1, t2, t3, tS}.(α11 α12 α13 θ1; α21 α22 α23 θ2; α31 α32 α33 θ3; ϕ1 ϕ2 ϕ3 Ξ).{h1, h2, h3, hS}];
(ζ // m12→1 // m13→1) == (ζ // m23→2 // m12→1)
True
```

## R3

... divide and conquer!

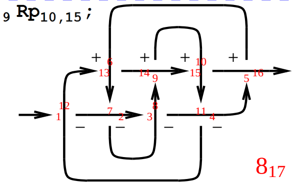
```
{Rm51 Rm62 Rp34 // m14→1 // m25→2 // m36→3,
 Rp61 Rm24 Rm35 // m14→1 // m25→2 // m36→3}
```

$$\left\{\begin{pmatrix}1 & h_1 & h_2 & h_3\\ t_1 & \frac{T_3}{T_2} & 0 & 0\\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0\\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1\end{pmatrix}, \begin{pmatrix}1 & h_1 & h_2 & h_3\\ t_1 & \frac{T_3}{T_2} & 0 & 0\\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0\\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1\end{pmatrix}\right\}$$



```
z = Rm12,1 Rm27 Rm83 Rm4,11 Rp16,5 Rp6,13 Rp14,9 Rp10,15;
Do[z = z // m1k→1, {k, 2, 16}];
z
```

$$\begin{pmatrix}11 - \frac{1}{T_1^3} + \frac{4}{T_1^2} - \frac{8}{T_1} - 8T_1 + 4T_1^2 - T_1^3 & h_1\\ t_1 & 1\end{pmatrix}$$



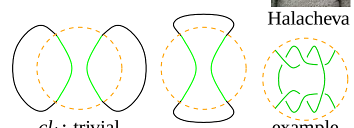
$8_{17}$

## Closed Components.

The Halacheva trace $tr_c$ satisfies $m_c^{ab} /\!/ tr_c = m_c^{ba} /\!/ tr_c$ and computes the MVA for all links in the atlas, but its domain is not understood:

$$\begin{array}{c|cc}\omega & c & S\\ \hline c & \alpha & \theta\\ S & \psi & \Xi\end{array} \xrightarrow[\mu := 1-\alpha]{tr_c} \begin{array}{c|c}\mu\omega & S\\ \hline S & \Xi + \psi\theta/\mu\end{array}$$

```
tr_c_[Γ[ω_, λ_]] := Module[{α, θ, ψ, Ξ},
  (α θ; ψ Ξ) = (∂_{t_c,h_c}λ ∂_{t_c}λ; ∂_{h_c}λ λ) /. (t | h)_c → 0;
  Γ[ω (1 - α), Ξ + ψ * θ / (1 - α)] // ΓCollect];
(ζ // m12→1 // tr1) == (ζ // m21→1 // tr1)
True
```



$cl_1$: trivial, $cl_2$: ribbon, example

Halacheva

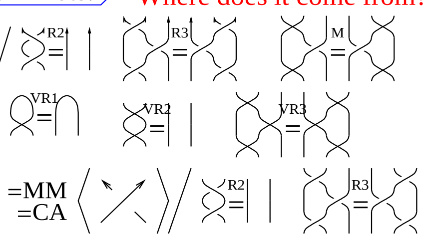
## Weaknesses.

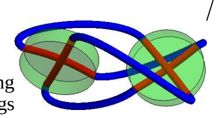
- $m_c^{ab}$ and $tr_c$ are non-linear.
- The product $\omega A$ is always Laurent, but my current proof takes induction with exponentially many conditions.
- I still don't understand $tr_c$, "unitarity", the algebra for ribbon knots.

## Where does it come from?

## v-Tangles.

$v\mathcal{T}$ :=PA ⟨ ... ⟩ / (R2, R3, M, VR1, VR2, VR3)

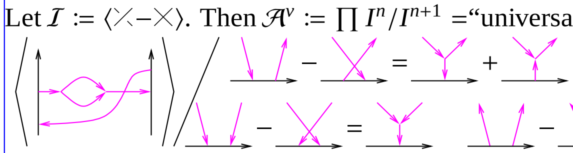


flying pogs

=MM
=CA

Let $\mathcal{I} := \langle \times - \times \rangle$. Then $\mathcal{A}^v := \prod I^n/I^{n+1}$ ="universal $\mathcal{U}(D\mathfrak{g})^{\otimes S}$"=



(Also IHX)

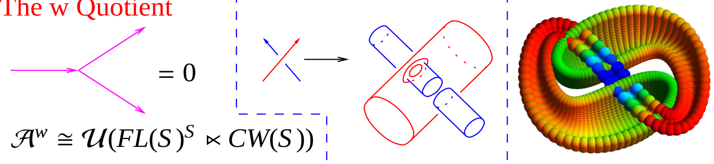
Fine print: No sources no sinks, AS vertices, internally acyclic, deg = (#vertices)/2.

## Likely Theorem.

[EK, En] There exists a homomorphic expansion (universal finite type invariant) $Z \colon v\mathcal{T} \to \mathcal{A}^v$. (issues suppressed)

Too hard! Let's look for "meta-monoid" quotients.

## The w Quotient



$= 0$

$\mathcal{A}^w \cong \mathcal{U}(FL(S)^S \ltimes CW(S))$

Video and more at http://www.math.toronto.edu/~drorbn/Talks/LesDiablerets-1508/