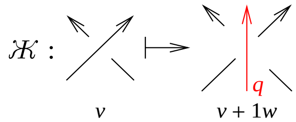
Crossing the Crossings

Abstract. The subject will be very close to Manturov's representation of vB_n into $\mathrm{Aut}(FG_{n+1})$ — I'll describe how I think about it in terms of a very simple minded map $\mathcal{X}$ from n-component v-tangles to $(n+1)$-component w-tangles. It is possible that you all know this already. Possibly my talk will be very short — it will be as long as it is necessary to describe $\mathcal{X}$ and say a few more words, and if this is little, so be it.

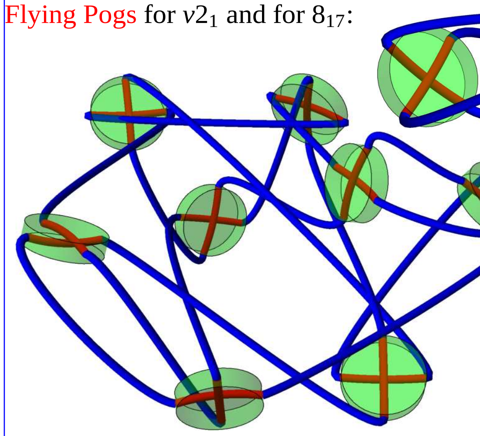
All you need is $\mathcal{X}$... • What is its domain? • What is its target? • Why should one care?



Virtual Knots. Virtual knots are the algebraic structure underlying the Reidemeister presentation of ordinary knots, without the topology. Locally they are knot diagrams modulo the Reidemeister relations; globally, who cares? So,

$$v\mathcal{T} = \mathrm{CA}\langle \text{⤱}, \text{⤲} : R1, R2, R3 \rangle \qquad \mathrm{CA} = \text{"Circuit Algebra"}$$

Flying Pogs for $v2_1$ and for 8_{17}:



Kauffman

Manturov

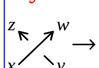
Boden

No! Note that also (with PA ="Planar Algebra")

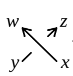
$$v\mathcal{T} = \mathrm{PA}\langle \text{⤱}, \text{⤲}, \times : R1, R2, R3, VR1, VR2, VR3, M \rangle,$$

but I have a prejudice, or a deeply held belief, that this is morally wrong!

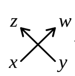
My moment of reckoning. Manturov's $VG(K)$: [Ma, BGHNW]



$\to$ $z = xyx^{-1}$, $w = x$



$\to$ $z = x^{-1}yx$, $w = x$



$\to$ $z = q^{-1}yq$, $w = qxq^{-1}$

Manturov's $\mu\colon vB_n \to \mathrm{Aut}(F(x_1,\ldots,x_n,q))$: [Ma, BGHNW]

$$\sigma_i = \text{⤱}_i \mapsto \begin{cases} x_i \mapsto x_i x_{i+1} x_i^{-1} \\ x_{i+1} \mapsto x_i \end{cases} \qquad \tau_i = \times_i \mapsto \begin{cases} x_i \mapsto q x_{i+1} q^{-1} \\ x_{i+1} \mapsto q^{-1} x_i q \end{cases}.$$

Easy resolution. Setting $y_i := q^i x_i q^{-i}$, we find that μ is equivalent to

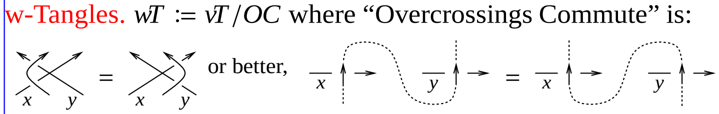
$$\text{⤱}_i \mapsto \begin{cases} y_i \mapsto y_i q^{-1} y_{i+1} q y_i^{-1} \\ y_{i+1} \mapsto q y_i q^{-1} \end{cases} \qquad \times_i \mapsto \begin{cases} y_i \mapsto y_{i+1} \\ y_{i+1} \mapsto y_i \end{cases},$$

and to me, virtual braids are anyways always pure. So really,

$$\sigma_{ij} \mapsto \begin{cases} y_i \mapsto q y_i q^{-1} \\ y_j \mapsto y_i^{-1} q^{-1} y_j q y_i \end{cases}.$$

But why does it exist? Especially, wherefore $vB_n \to wB_{n+1}$?

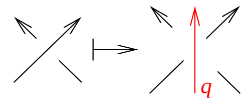
w-Tangles. $w\mathcal{T} := v\mathcal{T}/OC$ where "Overcrossings Commute" is:



π_1 is defined on $w\mathcal{T}$; Artin's representation ϕ is defined on wB_n.

Back to $\mathcal{X}$. The "crossing the crossings" map $\mathcal{X}\colon v\mathcal{T}_n \to w\mathcal{T}_{n+1}$ is defined by the picture below. Equally well, it is $\mathcal{X}\colon vB_n \to wB_{n+1}$. Better, it is $\mathcal{X}\colon v\mathcal{T}_n \to (nv+1w)\mathcal{T}$ or $\mathcal{X}\colon vB_n \to (nv+1w)B$.

Claims.

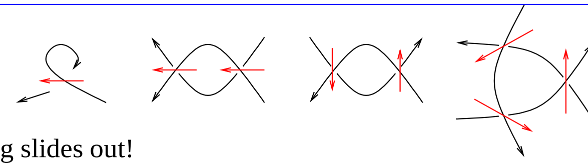


1. $\mathcal{X}$ is well defined.
2. On u-links, $\mathcal{X}$ "factors".
3. $\mathcal{X}$ does not respect OC.
4. $\mathcal{X}$ recovers Manturov's VG and μ: $VG(K) = \pi_1(\mathcal{X}(K))$, $\mu = \mathcal{X} \circ \phi = \phi /\!\!/ \mathcal{X}$.

Even better, $\mathcal{X}$ pulls back *any* invariant of 2-component w-knots to an invariant of virtual knots. in particular, there is a wheel-valued "non-commutative" invariant ω as in [BN] and DBN: Talks: Hamilton-1412 (next page).

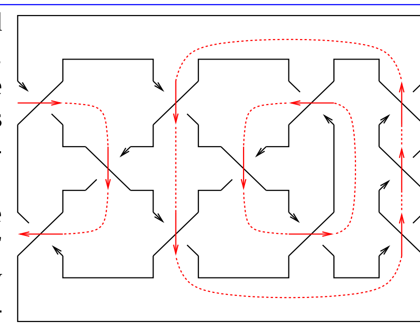
Likely, the various "2-variable Alexander polynomials" for virtual knots arise in this way.

Proof of 1.



Everything slides out!

Proof of 2. The net "red flow" into every face is 0, so the red arrows can be paired. They form cycles that can hover off the picture.



No proof of 3. Well, there simply is no proof that OC is respected, and it's easy to come up with counterexamples.

Proof of 4. A simple verification, except my conventions are off...

References.

[BN] D. Bar-Natan, *Balloons and Hoops and their Universal Finite Type Invariant, BF Theory, and an Ultimate Alexander Invariant,* Acta Mathematica Vietnamica **40-2** (2015) 271–329, arXiv:1308.1721.

[BGHNW] H. U. Boden, A. I. Gaudreau, E. Harper, A. J. Nicas, and L. White, *Virtual Knot Groups and Almost Classical Knots*, arXiv:1506.01726.

[Ma] V. O. Manturov, *On Invariants of Virtual Links*, Acta Applicandae Mathematica **72-3** (2002) 295–309.

Prejudices should always be re-evaluated!