

Work in Progress!

The Brute and the Hidden Paradise

Local Algebra (with van der Veen) Much can be reformulated as (non-standard) “quantum algebra” for the 4D Lie algebra $\mathfrak{g} = \langle b, c, u, w \rangle$ over $\mathbb{Q}[\epsilon]/(\epsilon^2 = 0)$, with b central and $[w, c] = w$, $[c, u] = u$, and $[u, w] = b - 2\epsilon c$. The key: $a_{ij} = (b_i - \epsilon c_i)c_j + u_i w_j$ in $\mathcal{U}(\mathfrak{g})^{\otimes(i,j)}$.



van der Veen

Some (new) representations of the (v-)braid groups.

oeβ/Reps

$B_{i,j}[\xi] := \xi / . v_j \mapsto (1-t) v_i + t v_j$

Burau (old)

Column@{lhs = {v1, v2, v3} // B1,2 // B1,3 // B2,3, rhs = {v1, v2, v3} // B2,3 // B1,3 // B1,2, lhs - rhs // Expand}

... testing R3

{v1, (1-t) v1 + t v2, (1-t) v1 + t ((1-t) v2 + t v3)}
{v1, (1-t) v1 + t v2,
(1-t) ((1-t) v1 + t v2) + t ((1-t) v1 + t v3)}
{0, 0, 0}

$G_{i,j}[\xi] := \xi / . v_j \mapsto (1-t_i) v_i + t_i v_j$

Gassner (old)

... Overcrossings Commute (OC):

Column@{lhs = {v1, v2, v3} // G1,2 // G1,3, Expand[lhs - ({v1, v2, v3} // G1,3 // G1,2)]}

... Undercrossings Commute (UC):

Column@{lhs = {v1, v2, v3} // G1,3 // G2,3, rhs = {v1, v2, v3} // G2,3 // G1,3, lhs - rhs // Expand}

{v1, v2, (1-t1) v1 + t1 ((1-t2) v2 + t2 v3)}
{v1, v2, (1-t2) v2 + t2 ((1-t1) v1 + t1 v3)}
{0, 0, v1 - t1 v1 - t2 v1 + t1 t2 v1 - v2 + t1 v2 + t2 v2 - t1 t2 v2}

Gassner Plus (new?)

$GP_{i,j}[\xi] := \text{Expand}[\xi / . \{u_j \mapsto (1-t_i) u_i + t_i u_j, f \cdot v_j \mapsto f(1-t_i) v_i + f t_i v_j + (t_i - 1)(t_i \partial_{t_i} f - t_j \partial_{t_j} f) u_i + f t_i u_i\}]$;

bas = {f[t1, t2, t3] v1, f[t1, t2, t3] v2, f[t1, t2, t3] v3, u1, u2, u3};

Short[lhs = bas // GP1,2 // GP1,3 // GP2,3, 2]

... R3 (left)

{f[t1, t2, t3] v1, f[t1, t2, t3] t1 u1 + f[t1, t2, t3] v1 - f[t1, t2, t3] t1 v1 + <<6>> + t1^2 u1 f^{(1,0,0)}[t1, t2, t3], <<1>> + <<19>> + <<1>>, <<1>>, u1 - t1 u1 + t1 u2, u1 - t1 u1 + t1 u2 - t1 t2 u2 + t1 t2 u3}

(bas // GP2,3 // GP1,3 // GP1,2) - lhs

... R3 (rest)

{0, 0, 0, 0, 0, 0}

(bas // GP1,2 // GP1,3) - (bas // GP1,3 // GP1,2)

... OC

{0, 0, 0, 0, 0, 0}

Question. Does Gassner Plus factor through Gassner?

$K\delta_{i,j} := \text{KroneckerDelta}[i, j]$;

Turbo-Gassner (new!)

$TG_{i,j}[\xi] := \text{Expand}[\xi / . \{f \cdot v_k \mapsto \text{Plus}[f v_k / . v_j \mapsto (1-t_i) v_i + t_i v_j, (1-t_i^{-1})(t_i \partial_{t_i} f - t_j \partial_{t_j} f) * (u_k / . u_j \mapsto (1-t_i) u_i + t_i u_j) * u_i w_j, K\delta_{k,i} f (u_j - u_i) u_i w_j, u_j \mapsto (1-t_i) u_i + t_i u_j, w_i \mapsto w_i + (1-t_i^{-1}) w_j, w_j \mapsto t_i^{-1} w_j\}]$;

bas = {f[t1, t2, t3] v1, f[t1, t2, t3] v2, f[t1, t2, t3] v3, u1, u2, u3, w1, w2, w3};

Satisfies R3...

(bas // TG1,2 // TG1,3) - (bas // TG1,3 // TG1,2)

... OC

{0, -f[t1, t2, t3] u1 u2 w3 + f[t1, t2, t3] t1 u1 u2 w3 + f[t1, t2, t3] u1 u3 w3 - f[t1, t2, t3] t1 u1 u3 w3, -f[t1, t2, t3] u1 u2 w2 + f[t1, t2, t3] t1 u1 u2 w2 + f[t1, t2, t3] u1 u3 w2 - f[t1, t2, t3] t1 u1 u3 w2, 0, 0, 0, 0, 0, 0, 0}

$\eta / : \eta[i] = 0; \eta / : \eta[i] \eta[j] = 0;$

Turbo-Burau (new!)

$TB_{i,j}[\xi] :=$

Expand[$\xi / . \{$

$f \cdot v_k \mapsto \text{Plus}[f v_k / . v_j \mapsto (1-t-\eta[i]) v_i + (t+\eta[i]) v_j, (t-1)(\text{Coefficient}[f, \eta[i]] - \text{Coefficient}[f, \eta[j]]) * (u_k / . u_j \mapsto (1-t) u_i + t u_j) * u_i w_j, K\delta_{k,i} (f / . \eta \rightarrow 0) (u_j - u_i) u_i w_j, u_j \mapsto (1-t) u_i + t u_j, w_i \mapsto w_i + (1-t^{-1}) w_j, w_j \mapsto t^{-1} w_j\}]$;

ff = f0 + f1 η[1] + f2 η[2] + f3 η[3];

bas = {ff v1, ff v2, ff v3, u1^2 w1, u2^2 w2, u1, u2, u3, w1, w2, w3};

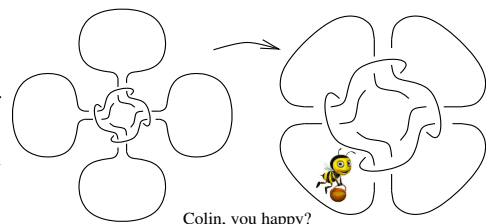
(bas // TB1,2 // TB1,3) - (bas // TB1,3 // TB1,2)

... OC

{0, -f0 u1 u2 w3 + t f0 u1 u2 w3 + f0 u1 u3 w3 - t f0 u1 u3 w3, -f0 u1 u2 w2 + t f0 u1 u2 w2 + f0 u1 u3 w2 - t f0 u1 u3 w2, 0, 0, 0, 0, 0, 0, 0}

Flower Surgery Theorem.

A knot is ribbon iff it is the result of n -petal flower surgery (from thin petals to wide petals) on an n -component unlink, for some n .



Colin, you happy?

References.

[BN] D. Bar-Natan, *Balloons and Hoops and their Universal Finite Type Invariant, BF Theory, and an Ultimate Alexander Invariant*, oeβ/KBH, arXiv: 1308.1721.

[BND] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects I, II, IV*, oeβ/WKO1, oeβ/WKO2, oeβ/WKO4, arXiv:1405.1956, arXiv: 1405.1955, arXiv:1511.05624.

[BNG] D. Bar-Natan and S. Garoufalidis, *On the Melvin-Morton-Rozansky conjecture*, Invent. Math. **125** (1996) 103–133.

[BNS] D. Bar-Natan and S. Selmani, *Meta-Monoids, Meta-Bicrossed Products, and the Alexander Polynomial*, J. of Knot Theory and its Ramifications **22-10** (2013), arXiv:1302.5689.

[En] B. Enriquez, *A Cohomological Construction of Quantization Functors of Lie Bialgebras*, Adv. in Math. **197-2** (2005) 430-479, arXiv:math/0212325.

[EK] P. Etingof and D. Kazhdan, *Quantization of Lie Bialgebras, I*, Selecta Mathematica **2** (1996) 1–41, arXiv:q-alg/9506005.

[GPV] M. Goussarov, M. Polyak, and O. Viro, *Finite type invariants of classical and virtual knots*, Topology **39** (2000) 1045–1068, arXiv: math.GT/9810073.

[Ha] A. Haviv, *Towards a diagrammatic analogue of the Reshetikhin-Turaev link invariants*, Hebrew University PhD thesis, Sep. 2002, arXiv: math.QA/0211031.

[MM] P. M. Melvin and H. R. Morton, *The coloured Jones function*, Commun. Math. Phys. **169** (1995) 501–520.

[PV] M. Polyak and O. Viro, *Gauss Diagram Formulas for Vassiliev Invariants*, Inter. Math. Res. Notices **11** (1994) 445–453.

[Ro] L. Rozansky, *A contribution of the trivial flat connection to the Jones polynomial and Witten’s invariant of 3d manifolds, I*, Comm. Math. Phys. **175-2** (1996) 275–296, arXiv:hep-th/9401061.

[Se] P. Ševera, *Quantization of Lie Bialgebras Revisited*, Sel. Math., NS, to appear, arXiv:1401.6164.



“God created the knots, all else in topology is the work of mortals.”

Leopold Kronecker (modified)



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The Knot Atlas