

**1-Smidgen  $sl_2$**  Let  $\mathfrak{g}_1$  be the 4-dimensional Lie algebra  $\mathfrak{g}_1 = \langle b, c, u, w \rangle$  over the ring  $R = \mathbb{Q}[\epsilon]/(\epsilon^2 = 0)$ , with  $b$  central and with  $[w, c] = w$ ,  $[c, u] = u$ , and  $[u, w] = b - 2\epsilon c$ , with CYBE  $r_{ij} = (b_i - \epsilon c_i)c_j + u_i w_j$  in  $\mathcal{U}(\mathfrak{g}_1)^{\otimes(i,j)}$ . Over  $\mathbb{Q}$ ,  $\mathfrak{g}_1$  is a **solvable approximation of  $sl_2$** :  $\mathfrak{g}_1 \supset \langle b, u, w, \epsilon b, \epsilon c, \epsilon u, \epsilon w \rangle \supset \langle b, \epsilon b, \epsilon c, \epsilon u, \epsilon w \rangle \supset 0$ . (note:  $\deg(b, c, u, w, \epsilon) = (1, 0, 1, 0, 1)$ )

**0-Smidgen  $sl_2 \odot$** . Let  $\mathfrak{g}_0$  be  $\mathfrak{g}_1$  at  $\epsilon = 0$ , or  $\mathbb{Q}\langle b, c, u, w \rangle / ([b, \cdot] = 0, [c, u] = u, [c, w] = -w, [u, w] = b$  with  $r_{ij} = b_i c_j + u_i w_j$ . It is  $\mathfrak{b}^* \rtimes \mathfrak{b}$  where  $\mathfrak{b}$  is the 2D Lie algebra  $\mathbb{Q}\langle c, w \rangle$  and  $(b, u)$  is the dual basis of  $(c, w)$ . For topology, it is more valuable than  $\mathfrak{g}_1 / sl_2$ , but topology already got by other means almost everything  $\mathfrak{g}_0$  gives.

**How did these arise?**  $sl_2 = \mathfrak{b}^+ \oplus \mathfrak{b}^- / \mathfrak{h} =: sl_2^+ / \mathfrak{h}$ , where  $\mathfrak{b}^+ = \langle c, w \rangle / [w, c] = w$  is a Lie bialgebra with  $\delta: \mathfrak{b}^+ \rightarrow \mathfrak{b}^+ \otimes \mathfrak{b}^+$  by  $\delta: (c, w) \mapsto (0, c \wedge w)$ . Going back,  $sl_2^+ = \mathcal{D}(\mathfrak{b}^+) = (\mathfrak{b}^+)^* \oplus \mathfrak{b}^+ = \langle b, u, c, w \rangle / \dots$ . **Idea.** Replace  $\delta \rightarrow \epsilon \delta$  over  $\mathbb{Q}[\epsilon]/(\epsilon^{k+1} = 0)$ . At  $k = 0$ , get  $\mathfrak{g}_0$ . At  $k = 1$ , get  $[w, c] = w$ ,  $[w, b'] = -\epsilon w$ ,  $[c, u] = u$ ,  $[b', u] = -\epsilon u$ ,  $[b', c] = 0$ , and  $[u, w] = b' - \epsilon c$ . Now note that  $b' + \epsilon c$  is central, so switch to  $b := b' + \epsilon c$ . This is  $\mathfrak{g}_1$ .

**Ordering Symbols.**  $\odot$  (*poly* | *specs*) plants the variables of *poly* in  $\mathcal{S}(\otimes \mathfrak{g}_i)$  on several tensor copies of  $\mathcal{U}(\mathfrak{g})$  according to *specs*. E.g.,  $\odot(c_1^3 u_1 c_2 e^{u_3} w_3^9 | x: w_3 c_1, y: u_1 u_3 c_2) = w^9 c^3 \otimes u e^u c \in \mathcal{U}(\mathfrak{g})_x \otimes \mathcal{U}(\mathfrak{g})_y$ . This enables the description of elements of  $\hat{\mathcal{U}}(\mathfrak{g})^{\otimes S}$  using commutative polynomials / power series.

**0-Smidgen Invariants.**  $r = Id \in \mathfrak{b}^- \otimes \mathfrak{b}^+$  solves the CYBE  $[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$  in  $\mathcal{U}(\mathfrak{g}_0)^{\otimes 3}$  and, by luck,

$$\begin{array}{c} \nearrow \\ + \\ i \end{array} \begin{array}{c} \searrow \\ - \\ j \end{array} = \begin{array}{c} \uparrow \\ + \\ i \end{array} \begin{array}{c} \uparrow \\ + \\ j \end{array} = R_{ij} = e^{r_{ij}} = e^{b_i c_j + u_i w_j} \in \mathcal{U}(\mathfrak{g}_{0,i} \oplus \mathfrak{g}_{0,j})$$

solves YB/R3.

**Lemma.**  $R_{ij} = e^{b_i c_j + u_i w_j} = \odot(\exp(b_i c_j + \frac{e^{b_i} - 1}{b_i} u_i w_j) | i: u_i, j: c_j w_j)$

**Example.**  $Z(T_0) = \sum_{m,n} \frac{b_i^{m-n} (e^{b_i} - 1)^n}{m! n!} u^n \otimes c^m w^n$ .

$$\odot(\exp(b_5 c_1 + \frac{e^{b_5} - 1}{b_5} u_5 w_1 + b_2 c_4 + \frac{e^{b_2} - 1}{b_2} u_2 w_4 - b_3 c_6 + \frac{e^{-b_3} - 1}{b_3} u_3 w_6) | \text{“ucw form”} \\ x: c_1 w_1 u_2, y: u_3 c_4 w_4 u_5 c_6 w_6) = \odot(\zeta | x: u_x c_x w_x, y: u_y c_y w_y)$$

**Goal.** Write  $\zeta$  as a Gaussian:  $\omega e^{L+Q}$  where  $L$  bilinear in  $b_i$  and  $c_i$  with integer coefficients,  $Q$  a balanced quadratic in  $u_i$  and  $w_i$  with coefficients in  $R_S := \mathbb{Q}(b_i, e^{b_i})$ , and  $\omega \in R_S$ .

**The Big  $\mathfrak{g}_0$  Lemma.** Under  $[c, u] = u$ ,  $[c, w] = -w$ , and  $[u, w] = b$ :

- 1a.  $N^{cu} := \odot(e^{\gamma c + \beta u} | uc) \xrightarrow{\gamma} \odot(e^{\gamma c + e^{\beta} \beta u} | cu)$  (means  $e^{\beta u} e^{\gamma c} = e^{\gamma c} e^{\beta u}$ )
- 1b.  $N^{wc} := \odot(e^{\gamma c + \alpha w} | wc) \xrightarrow{\gamma} \odot(e^{\gamma c + e^{\alpha} \alpha w} | cw)$  ... in the  $(ax + b)$  group)
2.  $\odot(e^{\alpha w + \beta u} | wu) = \odot(e^{-b\alpha\beta + \alpha w + \beta u} | uw)$  (the Weyl relations)
3.  $\odot(e^{\delta u w} | wu) e^{\beta u} = e^{\gamma \beta u} \odot(e^{\delta u w} | wu)$ , with  $\gamma = (1 + b\delta)^{-1}$
- (a. expand and crunch. b. use  $w = b\hat{x}$ ,  $u = \partial_x$ . c. use “scatter and glow”.)
4.  $\odot(e^{\delta u w} | wu) = \odot(\gamma e^{\gamma \delta u w} | uw)$  (same techniques)
5.  $N^{wu} := \odot(e^{\beta u + \alpha w + \delta u w} | wu) \xrightarrow{\beta} \odot(\gamma e^{-b\alpha\beta + \gamma\alpha w + \gamma\beta u + \gamma\delta u w} | uw)$
6.  $N_k^{c_i c_j} := \odot(\zeta | c_i c_j) \xrightarrow{\gamma} \odot(\zeta / (c_i, c_j \rightarrow c_k) | c_k)$

**Sneaky.**  $\alpha$  may contain (other)  $u$ 's,  $\beta$  may contain (other)  $w$ 's.

**Strand Stitching.**  $m_k^{ij}$  is defined as the composition

$$u_i c_i \overline{w_i u_j} c_j w_j \xrightarrow{N_x^{w_i u_j}} u_i \overline{c_i u_x} \overline{w_x c_j} w_j \xrightarrow{N_x^{c_i u_x} // N_x^{w_x c_j}} \overline{u_i u_x} \overline{c_x c_x} \overline{w_x w_j} \\ \xrightarrow{i,j,x \rightarrow k} u_k c_k w_k$$

**On to 1-smidgen invariants**, where much is the same. . .

**The Big  $\mathfrak{g}_1$  Lemma.** Parts 1 and 6 are the same, yet  
5.  $\odot(e^{\alpha w + \beta u + \delta u w} | wu) = \odot(\gamma(1 + \epsilon v \Lambda) e^{\gamma(-b\alpha\beta + \alpha w + \beta u + \delta u w)} | ucw)$   
Here  $\Lambda$  is for  $\Lambda\acute{o}\gamma\omicron\varsigma$ , “a principle of order and knowledge”, a balanced quartic in  $\alpha, \beta, u, c$ , and  $w$ :

$$\begin{aligned} \Lambda = & -b\gamma(\alpha^2 \beta^2 \gamma^2 + 4\alpha\beta\delta\gamma + 2\delta^2)/2 + \beta^2 \delta \gamma^3 (b\delta + 2)u^2/2 \\ & + \delta^3 \gamma^3 (3b\delta + 4)u^2 w^2/2 + \beta \delta^2 \gamma^3 (2b\delta + 3)u^2 w \\ & + \alpha \delta^2 \gamma^3 (2b\delta + 3)u w^2 + 2\delta \gamma^2 (b\delta + 2)(\alpha\beta\gamma + \delta)u w \\ & + \alpha^2 \delta \gamma^3 (b\delta + 2)w^2/2 + 2(\alpha\beta\gamma + \delta)c + 2\beta\delta\gamma u c + 2\delta^2 \gamma u c w \\ & + 2\alpha\delta\gamma c w + \beta\gamma^2 (\alpha\beta\gamma + 2\delta)u + \alpha\gamma^2 (\alpha\beta\gamma + 2\delta)w. \end{aligned}$$

**Proof.** A lengthy computation. (Verification:  $\omega\epsilon\beta/\text{Big}$ )

**Problem.** We now need to normal-order perturbed Gaussians!

**Solution.** Borrow some tactics from QFT:

$$\odot(\epsilon P(c, u) e^{\gamma c + \beta u} | uc) = \odot(\epsilon P(\partial_\gamma, \partial_\beta) e^{\gamma c + \beta u} | uc) = \odot(\epsilon P(\partial_\gamma, \partial_\beta) e^{\gamma c + e^{-\gamma} \beta u} | cu),$$

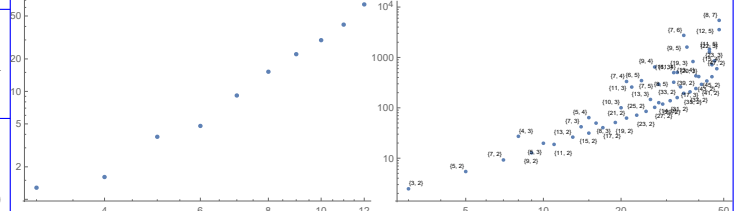
$$\odot(\epsilon P(u, w) e^{\alpha w + \beta u + \delta u w} | wu) = \odot(\epsilon P(\partial_\beta, \partial_\alpha) \gamma e^{\gamma(-b\alpha\beta + \alpha w + \beta u + \delta u w)} | ucw)$$

**Finally,** the values of the generators  $\nearrow, \searrow, \vec{n}$ , and  $\underline{u}$ , are set by solving many equations, non-uniquely.

**Pragmatic Simplifications.** Set  $t := e^b$ , work with  $v := (t-1)u/b$ , and set  $\mathbb{E}(\omega, L, Q, P) := \odot(\omega^{-1} e^{L+Q} \omega (1 + \epsilon \omega^{-4} P) : (i: v_i c_i w_i))$ . Now  $\omega \in R_S := \mathbb{Z}[t_i, t_i^{-1}]$  is Laurent,  $L = \sum l_{ij} \log(t_i) c_j$  with  $l_{ij} \in \mathbb{Z}$ ,  $Q = \sum q_{ij} v_i w_j$  with  $q_{ij} \in R_S$ , and  $P$  is a quartic polynomial in  $v_i, c_j, w_k$  with coefficients in  $R_S$ . The operations are lightly modified, and the  $\Lambda\acute{o}\gamma\omicron\varsigma$  and the values of the generators become somewhat simpler, as in the implementation below.

**Rough complexity estimate,** after  $t_k \rightarrow t$ :  $n$ : xing  $\frac{n}{A} \sum_{d=0}^4 \frac{w^{4-d} w^d n^2}{E F G} = n^3 w^4 \in [n^5, n^7]$  number;  $w$ : width, maybe  $\sim \sqrt{n}$ .  $A$ : go over stitchings in order.  $B$ : multiplication ops per  $N^{u_i w_j}$ .  $d$ : deg of  $u_i, w_j$  in  $P$ .  $E$ : #terms of deg  $d$  in  $P$ .  $F$ : ops per term.  $G$ : cost per polynomial multiplication op.

**Experimental Analysis ( $\omega\epsilon\beta/\text{Exp}$ ).** Log-log plots of computation time (sec) vs. crossing number, for all knots with up to 12 crossings (mean times) and for all torus knots with up to 48 crossings:



**Conjecture** (checked on the same collections). Given a knot  $K$  with Alexander polynomial  $A$ , there is a polynomial  $\rho_1$  such that

$$P = A^2 \frac{(t-1)^3 \rho_1 + t^2 (2vw + (1-t)(1-2c)) AA'}{(1-t)t}$$

Furthermore,  $A$  and  $\rho_1$  are symmetric under  $t \rightarrow t^{-1}$ , so let  $A^+$  and  $\rho_1^+$  be their “positive parts”, so e.g.,  $\rho_1(t) = \rho_1^+(t) + \rho_1^+(t^{-1}) - \rho_1^+(0)$ .

**Power.** On the 250 knots with at most 10 crossings, the pair  $(A, \rho_1)$  attains 250 distinct values, while (Khovanov, HOMFLY-PT) attains only 249 distinct values. To 11 crossings the numbers are (802, 788, 772) and to 12 they are (2978, 2883, 2786).

**Genus.** Up to 12 xings, always  $\deg \rho_1^+ \leq 2g - 1$ , where  $g$  is the 3-genus of  $K$  (equality for 2530 knots). This gives a lower bound on  $g$  in terms of  $\rho_1$  (conjectural, but undoubtedly true). This bound is often weaker than the Alexander bound, yet for 10 of the 12-xing Alexander failures it does give the right answer.