Dror Bar-Natan: Talks: LesDiablerets-1708:

The Dogma is Wrong

Follows Rozansky [Ro1, Ro2, Ro3] and Overbay [Ov], joint with van der Veen. Preliminary writeup [BV1], fuller writeup [BV2]. More at ωεβ/talks.

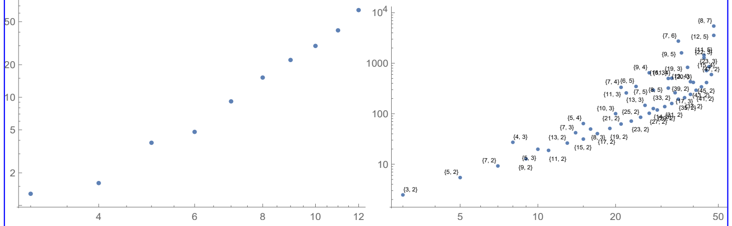
Happy Birthday Anton!

ωεβ:=http://drorbn.net/ld17/

Abstract. It has long been known that there are knot invariants associated to semi-simple Lie algebras, and there has long been a dogma as for how to extract them: "quantize and use representation theory". We present an alternative and better procedure: "centrally extend, approximate by solvable, and learn how to re-order exponentials in a universal enveloping algebra". While equivalent to the old invariants via a complicated process, our invariants are in practice stronger, faster to compute (poly-time vs. exp-time), and clearly carry topological information.

KiW 43 Abstract (ωεβ/kiw). Whether or not you like the formulas on this page, they describe the strongest truly computable knot invariant we know.

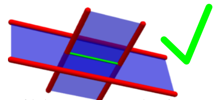
Experimental Analysis (ωεβ/Exp). Log-log plots of computation time (sec) vs. crossing number, for all knots with up to 12 crossings (mean times) and for all torus knots with up to 48 crossings:



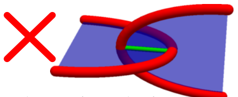
Power. On the 250 knots with at most 10 crossings, the pair (ω,ρ_1) attains 250 distinct values, while (Khovanov, HOMFLY-PT) attains only 249 distinct values. To 11 crossings the numbers are (802, 788, 772) and to 12 they are (2978, 2883, 2786).

Genus. Up to 12 xings, always ρ_1 is symmetric under $t \leftrightarrow t^{-1}$. With ρ_1^+ denoting the positive-degree part of ρ_1, always $\deg\rho_1^+ \leq 2g-1$, where g is the 3-genus of K (equality for 2530 knots). This gives a lower bound on g in terms of ρ_1 (conjectural, but undoubtedly true). This bound is often weaker than the Alexander bound, yet for 10 of the 12-xing Alexander failures it does give the right answer.

Ribbon Knots.

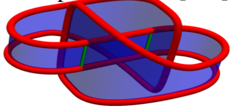


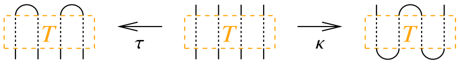
a ribbon singularity



a clasp singularity

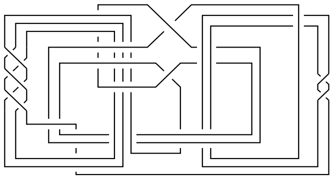
example [BN]





Gompf, Scharlemann, Thompson [GST]

$$\begin{tikzcd}
 & U\in\mathcal{T}_n & & 1\in\mathcal{A}_n \\
\mathcal{T}_{2n} \arrow[ur, "\tau"] \arrow[dr, "\kappa"'] \arrow[rr, "z"] & & \mathcal{A}_{2n} \arrow[ur, "\tau"] \arrow[dr, "\kappa"'] & \\
 & \text{ribbon } K\in\mathcal{T}_1 & & z(K)\in\mathcal{R}\subseteq\mathcal{A}_1
\end{tikzcd}$$



[Vo]: Works for Alexander!

with $\mathcal{R} := \kappa(\tau^{-1}(1))$

$$A^+ = -t^8+2t^7-t^6-2t^4+5t^3-2t^2-7t+13$$

$$\rho_1^+ = 5t^{15}-18t^{14}+33t^{13}-32t^{12}+2t^{11}+42t^{10}-62t^9-8t^8+166t^7-242t^6+108t^5+132t^4-226t^3+148t^2-11t-36$$

Faster is better, leaner is meaner!

Ordering Symbols. $\mathbb{O}(poly\mid specs)$ plants the variables of *poly* in $\mathcal{S}(\oplus_i\mathfrak{g})$ on several tensor copies of $\mathcal{U}(\mathfrak{g})$ according to *specs*. E.g.,

$$\mathbb{O}\left(a_1^3y_1a_2e^{y_3}x_3^9\mid x_3a_1\otimes y_1y_3a_2\right) = x^9a^3\otimes ye^ya\in\mathcal{U}(\mathfrak{g})\otimes\mathcal{U}(\mathfrak{g})$$

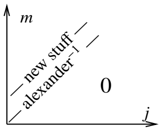
This enables the description of elements of $\hat{\mathcal{U}}(\mathfrak{g})^{\otimes S}$ using commutative polynomials / power series.

Theorem ([BNG], conjectured [MM], elucidated [Ro1]). Let $J_d(K)$ be the coloured Jones polynomial of K, in the d-dimensional representation of sl_2. Writing

Melvin, Morton, Garoufalidis

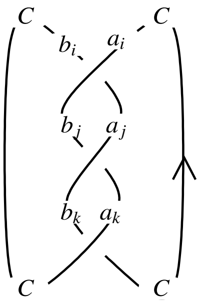
$$\left.\frac{(q^{1/2}-q^{-1/2})J_d(K)}{q^{d/2}-q^{-d/2}}\right|_{q=e^\hbar} = \sum_{j,m\geq 0} a_{jm}(K)d^j\hbar^m,$$

"below diagonal" coefficients vanish, $a_{jm}(K)=0$ if $j>m$, and "on diagonal" coefficients give the inverse of the Alexander polynomial: $\left(\sum_{m=0}^\infty a_{mm}(K)\hbar^m\right)\cdot\omega(K)(e^\hbar)=1$.



"Above diagonal" we have **Rozansky's Theorem** [Ro3, (1.2)]:

$$J_d(K)(q) = \frac{q^d-q^{-d}}{(q-q^{-1})\omega(K)(q^d)}\left(1+\sum_{k=1}^\infty\frac{(q-1)^k\rho_k(K)(q^d)}{\omega^{2k}(K)(q^d)}\right).$$



The Yang-Baxter Technique. Given an algebra U (typically $\hat{\mathcal{U}}(\mathfrak{g})$ or $\hat{\mathcal{U}}_q(\mathfrak{g})$) and elements

$$R=\sum a_i\otimes b_i\in U\otimes U \quad\text{and}\quad C\in U,$$

form

$$Z=\sum_{i,j,k} Ca_ib_ja_kC^2b_ia_jb_kC.$$

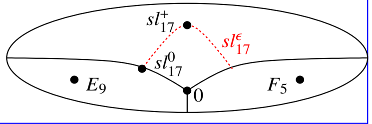
Problem. Extract information from Z.

The Dogma. Use representation theory. In principle finite, but *slow*.

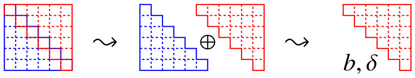
The Loyal Opposition. For certain algebras, work in a homomorphic poly-dimensional "space of formulas".

$$\begin{tikzcd}
\{\mathcal{F}_S\} \arrow[loop left, "m_k^{ij}"] \arrow[r, "\mathbb{E}"] & \{U^{\otimes S}\} \arrow[loop right, "m_k^{ij}"]
\end{tikzcd}$$

The (fake) moduli of Lie algebras on V, a quadratic variety in $(V^*)^{\otimes 2}\otimes V$ is on the right. We care about $sl_{17}^k := sl_{17}^\epsilon/(\epsilon^{k+1}=0)$.



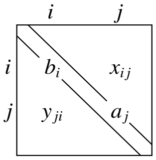
Recomposing gl_n. Half is enough! $gl_n\oplus\mathfrak{a}_n=\mathcal{D}(◹, b, \delta)$:



$b(◹) = b\colon ◹\otimes◹\to◹$

$b(◺)\rightsquigarrow\delta\colon ◹\to◹\otimes◹$

Now define $gl_n^\epsilon := \mathcal{D}(◹, b, \epsilon\delta)$. Schematically, this is $[◹,◹]=◹$, $[◺,◺]=\epsilon◺$, and $[◹,◺]=◺+\epsilon◹$. In detail, it is



$$[x_{ij},x_{kl}]=\delta_{jk}x_{il}-\delta_{li}x_{kj} \qquad [y_{ij},y_{kl}]=\epsilon\delta_{jk}y_{il}-\epsilon\delta_{li}y_{kj}$$

$$[x_{ij},y_{kl}]=\delta_{jk}(\epsilon\delta_{j<k}x_{il}+\delta_{il}(b_i+\epsilon a_i)/2+\delta_{i>l}y_{il}) -\delta_{li}(\epsilon\delta_{k<j}x_{kj}+\delta_{kj}(b_j+\epsilon a_j)/2+\delta_{k>j}y_{kj})$$

$$[a_i,x_{jk}]=(\delta_{ij}-\delta_{ik})x_{jk} \qquad [b_i,x_{jk}]=\epsilon(\delta_{ij}-\delta_{ik})x_{jk}$$

$$[a_i,y_{jk}]=(\delta_{ij}-\delta_{ik})y_{jk} \qquad [b_i,y_{jk}]=\epsilon(\delta_{ij}-\delta_{ik})y_{jk}$$

The Main sl_2 Theorem. Let $\mathfrak{g}^\epsilon=\langle t,y,a,x\rangle/([t,\cdot]=0, [a,x]=x, [a,y]=-y, [x,y]=t-2\epsilon a)$ and let $\mathfrak{g}_k=\mathfrak{g}^\epsilon/(\epsilon^{k+1}=0)$. The $\mathfrak{g}_k$-invariant of any S-component tangle K can be written in the form $Z(K)=\mathbb{O}\left(\omega\mathbb{e}^{L+Q+P}\colon\ \bigotimes_{i\in S}y_ia_ix_i\right)$, where ω is a scalar (a rational function in the variables t_i and their exponentials $T_i:=\mathbb{e}^{t_i}$), where $L=\sum l_{ij}t_ia_j$ is a quadratic in t_i and a_j with integer coefficients l_{ij}, where $Q=\sum q_{ij}y_ix_j$ is a quadratic in the variables y_i and x_j with scalar coefficients q_{ij}, and where P is a polynomial in $\{\epsilon, y_i, a_i, x_i\}$ (with scalar coefficients) whose ϵ^d-term is of degree at most $2d+2$ in $\{y_i, \sqrt{a_i}, x_i\}$. Furthermore, after setting $t_i=t$ and $T_i=T$ for all i, the invariant $Z(K)$ is poly-time computable.

Video and more at `http://www.math.toronto.edu/~drorbn/Talks/LesDiablerets-1708/`