

$$\mathbb{E} \left[a_3 \alpha_1 + a_3 \alpha_2 + t_3 (\tau_1 + \tau_2), \right. \\ \left. y_3 \eta_1 + e^{-\gamma \alpha_1} y_3 \eta_2 + e^{-\gamma \alpha_2} x_3 \xi_1 + \frac{(1 - T_3) \eta_2 \xi_1}{\hbar} + x_3 \xi_2, \right. \\ \left. 1 + \frac{1}{4 \hbar} \eta_2 \xi_1 (8 \hbar a_3 T_3 + 4 e^{-\gamma \alpha_1 - \gamma \alpha_2} \gamma \hbar^2 x_3 y_3 + 2 e^{-\gamma \alpha_1} \gamma \hbar y_3 \eta_2 - \right. \\ \left. 6 e^{-\gamma \alpha_1} \gamma \hbar T_3 y_3 \eta_2 + 2 e^{-\gamma \alpha_2} \gamma \hbar x_3 \xi_1 - 6 e^{-\gamma \alpha_2} \gamma \hbar T_3 x_3 \xi_1 + \right. \\ \left. \gamma \eta_2 \xi_1 - 4 \gamma T_3 \eta_2 \xi_1 + 3 \gamma T_3^2 \eta_2 \xi_1) \in + O[\epsilon]^2 \right]$$

```
S[U_, kk_] := S[U, kk] = Module[{OE},
  OE = m3,2,1->1[ExpQU1,$k[η, S1[QU[y1]]] /. QU -> Times]
  ExpQU2,$k[α, S2[QU[a2]]] /. QU -> Times]
  ExpQU3,$k[ξ, S3[QU[x3]]] /. QU -> Times];]
  E[-t1 τ1 + OE[[1]], OE[[2]], OE[[3]]] /.
  {η -> η1, α -> α1, ξ -> ξ1};]
ts_i := S[$U, $k] /. {(v : τ | η | α | ξ)1 -> v_i,
  (v : t | T | y | a | x)1 -> v_i};]
```

$$tS_1 \\ \mathbb{E} \left[-a_1 \alpha_1 - t_1 \tau_1, \right. \\ \left. \frac{-e^{\gamma \alpha_1} \hbar y_1 \eta_1 - e^{\gamma \alpha_1} \hbar T_1 x_1 \xi_1 + e^{\gamma \alpha_1} \eta_1 \xi_1 - e^{\gamma \alpha_1} T_1 \eta_1 \xi_1}{\hbar T_1}, 1 + \right. \\ \left. \frac{1}{4 \hbar T_1^2} (4 e^{\gamma \alpha_1} \gamma \hbar^2 T_1 y_1 \eta_1 - 4 e^{\gamma \alpha_1} \hbar^2 a_1 T_1 y_1 \eta_1 - 2 e^{2 \gamma \alpha_1} \gamma \hbar^2 y_1^2 \eta_1^2 - \right. \\ \left. 4 e^{\gamma \alpha_1} \hbar^2 a_1 T_1^2 x_1 \xi_1 - 4 e^{\gamma \alpha_1} \gamma \hbar T_1 \eta_1 \xi_1 + 8 e^{\gamma \alpha_1} \hbar a_1 T_1 \eta_1 \xi_1 + \right. \\ \left. 4 e^{\gamma \alpha_1} \gamma \hbar T_1^2 \eta_1 \xi_1 - 4 e^{2 \gamma \alpha_1} \gamma \hbar^2 T_1 x_1 y_1 \eta_1 \xi_1 + 6 e^{2 \gamma \alpha_1} \gamma \right. \\ \left. \hbar y_1 \eta_1^2 \xi_1 - 2 e^{2 \gamma \alpha_1} \gamma \hbar T_1 y_1 \eta_1^2 \xi_1 - 2 e^{2 \gamma \alpha_1} \gamma \hbar^2 T_1^2 x_1^2 \xi_1^2 + \right. \\ \left. 6 e^{2 \gamma \alpha_1} \gamma \hbar T_1 x_1 \eta_1 \xi_1^2 - 2 e^{2 \gamma \alpha_1} \gamma \hbar T_1^2 x_1 \eta_1 \xi_1^2 - 3 e^{2 \gamma \alpha_1} \gamma \eta_1^2 \xi_1^2 + \right. \\ \left. 4 e^{2 \gamma \alpha_1} \gamma T_1 \eta_1^2 \xi_1^2 - e^{2 \gamma \alpha_1} \gamma T_1^2 \eta_1^2 \xi_1^2) \in + O[\epsilon]^2 \right]$$

```
Δ[U_, kk_] := Δ[U, kk] = Module[{OE},
  OE = Block[{$k = kk, $p = kk + 1},
  m1,3,5->1@
  m2,4,6->2@Times[(* Warning:
  wrong unless $p>=$k+1! *)
  ReplacePart[1 -> 0]@
  ExpQU1,$k[η, Δ1->1,2[QU[y1]]] /. QU -> Times],
  ReplacePart[2 -> 0]@
  ExpQU3,$k[α, Δ3->3,4[QU[a3]]] /. QU -> Times],
  ReplacePart[1 -> 0]@
  ExpQU5,$k[ξ, Δ5->5,6[QU[x5]]] /. QU -> Times]
  ] /. {η -> η1, α -> α1, ξ -> ξ1};]
  E[t1 (t1 + t2) + α1 (a1 + a2), OE[[2]], OE[[3]]];]
tΔi->j,k_ :=
  Δ[$U, $k] /. {(v : τ | η | α | ξ)1 -> v_i,
  (v : t | T | y | a | x)1 -> v_j, (v : t | T | y | a | x)2 -> v_k};]
```

$$t\Delta_{1 \rightarrow 1, 2} \\ \mathbb{E} \left[(a_1 + a_2) \alpha_1 + (t_1 + t_2) \tau_1, y_1 \eta_1 + T_1 y_2 \eta_1 + x_1 \xi_1 + x_2 \xi_1, \right. \\ \left. 1 + \frac{1}{2} (-2 \hbar a_1 T_1 y_2 \eta_1 + \gamma \hbar T_1 y_1 y_2 \eta_1^2 - 2 \hbar a_1 x_2 \xi_1 + \gamma \hbar x_1 x_2 \xi_1^2) \in + \right. \\ \left. O[\epsilon]^2 \right]$$

The Faddeev-Quesne formula:

$$e_{q_-, k_-}[X_-] := e^{\sum_{j=1}^{k_-+1} \frac{(1-q)^j x^j}{j(1-q^j)}}; e_{q_-, k_-}[X_-] := e_{q_-, k_-}[X_-]$$

```
R[QU, kk_] :=
  R[QU, kk] = E[-\frac{\hbar a_2 t_1}{\gamma}, \hbar x_2 y_1,
  Series[e^{\hbar \gamma^{-1} t_1 a_2 - \hbar y_1 x_2}
  (e^{\hbar b_1 a_2} e_{q_{\hbar, kk}}[\hbar y_1 x_2] /. b_1 -> \gamma^{-1} (e a_1 - t_1)),
  {\epsilon, \theta, kk}]]];
tRi,j_ :=
  R[$U, $k] /. {(v : t | T | y | a | x)1 -> v_i,
  (v : t | T | y | a | x)2 -> v_j};]
tRi,j_ := tRi,j - B_j ~ tS_j;
{tR1,2, tR1,2}
{E[-\frac{\hbar a_2 t_1}{\gamma}, \hbar x_2 y_1, 1 + (\frac{\hbar a_1 a_2}{\gamma} - \frac{1}{4} \gamma \hbar^3 x_2^2 y_1^2) \in + O[\epsilon]^2],
  E[\frac{\hbar a_2 t_1}{\gamma}, -\frac{\hbar x_2 y_1}{T_1}, 1 + \frac{1}{4 \gamma T_1^2}
  (-4 \hbar a_1 a_2 T_1^2 - 4 \gamma \hbar^2 a_1 T_1 x_2 y_1 - 4 \gamma \hbar^2 a_2 T_1 x_2 y_1 - 3 \gamma^2 \hbar^3 x_2^2 y_1^2)
  \in + O[\epsilon]^2]}]
```

tC is the counterclockwise spinner; tC̄ is its inverse.

```
tC_i := E[0, \theta, T_i^{1/2} e^{-\epsilon a_i \hbar} + \theta_{\$k}];
tC_i := E[0, \theta, T_i^{-1/2} e^{\epsilon a_i \hbar} + \theta_{\$k}];
Block[{$k = 3}, {tC1, tC2}]
{E[0, \theta,
  \sqrt{T_1} - \hbar a_1 \sqrt{T_1} \in + \frac{1}{2} \hbar^2 a_1^2 \sqrt{T_1} \in^2 - \frac{1}{6} (\hbar^3 a_1^3 \sqrt{T_1}) \in^3 + O[\epsilon]^4],
  E[0, \theta, \frac{1}{\sqrt{T_2}} + \frac{\hbar a_2 \in}{\sqrt{T_2}} + \frac{\hbar^2 a_2^2 \in^2}{2 \sqrt{T_2}} + \frac{\hbar^3 a_2^3 \in^3}{6 \sqrt{T_2}} + O[\epsilon]^4]}]
```

```
Kink[QU, kk_] :=
  Kink[QU, kk] =
  Block[{$k = kk}, (tR1,3 tC2) ~ B1,2 ~ tm1,2->1 ~ B1,3 ~ tm1,3->1];]
tKink_i := Kink[$U, $k] /. {(v : t | T | y | a | x)1 -> v_i};]
Kink[QU, kk_] :=
  Kink[QU, kk] =
  Block[{$k = kk}, (tR1,3 tC2) ~ B1,2 ~ tm1,2->1 ~ B1,3 ~ tm1,3->1];]
tKink_i := Kink[$U, $k] /. {(v : t | T | y | a | x)1 -> v_i};]
```

Alternative Algorithms

```
λalt,k_ [CU] := If[k == 0, 1, Module[{eq, d, b, c, so},
  eq = ρ @ e^{\epsilon x u}.ρ @ e^{\eta y u} = ρ @ e^d y u.ρ @ e^c (t1 c u - 2 \epsilon a u).ρ @ e^b x u;
  {so} = Solve[Thread[Flatten[eq], {d, b, c}]] /.
  C@1 -> 0;
  Series[e^{-\eta y - \epsilon x + \eta \epsilon t + c t + d y - 2 \epsilon c a + b x} /. so, {\epsilon, \theta, k}]]];]
```

The Trefoil

```
Block[{$k = 1},
  Z = tR1,5 tR6,2 tR3,7 tC4 tKink8 tKink9 tKink10;
  Do[Z = Z ~ B1,k ~ tm1,k->1, {k, 2, 10}]; Z]
E[0, \theta, \frac{T_1}{1 - T_1 + T_1^2} +
  ((-2 \hbar a_1 T_1 - \gamma \hbar T_1^2 + 2 \hbar a_1 T_1^2 + 2 \gamma \hbar T_1^3 - 3 \gamma \hbar T_1^4 - 2 \hbar a_1 T_1^4 +
  2 \gamma \hbar T_1^5 + 2 \hbar a_1 T_1^5 - 2 \gamma \hbar^2 T_1 x_1 y_1 - 2 \gamma \hbar^2 T_1^4 x_1 y_1) \in) /
  (1 - 3 T_1 + 6 T_1^2 - 7 T_1^3 + 6 T_1^4 - 3 T_1^5 + T_1^6) + O[\epsilon]^2]
```

diagram	n'_k Alexander's ω^+ Today's / Rozansky's ρ^+	genus / ribbon unknotting number / amphicheiral	diagram	n'_k Alexander's ω^+ Today's / Rozansky's ρ^+	genus / ribbon unknotting number / amphicheiral
	0^q_1 1 0	0 / ✓ 0 / ✓		3^q_1 $t - 1$ t	1 / ✗ 1 / ✗
	4^q_1 $3 - t$ 0	1 / ✗ 1 / ✓		5^q_1 $t^2 - t + 1$ $2t^3 + 3t$	2 / ✗ 2 / ✗
	5^q_2 $2t - 3$ $5t - 4$	1 / ✗ 1 / ✗		6^q_1 $5 - 2t$ $t - 4$	1 / ✓ 1 / ✗