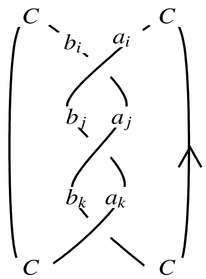


The Yang-Baxter Technique. Given an algebra U (typically $\hat{\mathcal{U}}(\mathfrak{g})$ or $\hat{\mathcal{U}}_q(\mathfrak{g})$) and elements

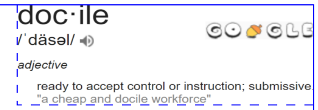
$$R = \sum a_i \otimes b_i \in U \otimes U \quad \text{and} \quad C \in U,$$

form

$$Z = \sum_{i,j,k} C a_i b_j a_k C^2 b_i a_j b_k C.$$

Problem. Extract information from Z.
The Dogma. Use representation theory. In principle finite, but *slow*.

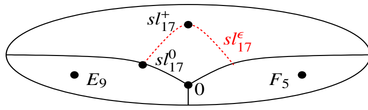
Definition. A "docile perturbed Gaussian" in the variables $(z_i)_{i\in S}$ over the ring R is an expression of the form



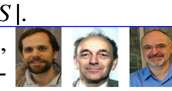
$$\mathbb{e}^{q^{ij}z_iz_j}P = \mathbb{e}^{q^{ij}z_iz_j}\left(\sum_{k\geq 0}\epsilon^k P_k\right),$$

where all coefficients are in R and where P is a "docile series": $\deg P_k \leq 4k$.
Docililty Matters! The rank of the space of docile series to ϵ^k is polynomial in the number of variables $|S|$.

The (fake) moduli of Lie algebras on V, a quadratic variety in $(V^*)^{\otimes 2}\otimes V$ is on the right. We care about $sl_{17}^k := sl_{17}^\epsilon/(\epsilon^{k+1}=0)$.

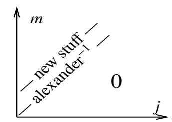


Theorem ([BNG], conjectured [MM], elucidated [Ro1]). Let $J_d(K)$ be the coloured Jones polynomial of K, in the d-dimensional representation of sl_2. Writing


Melvin, Morton, Garoufalidis

$$\left.\frac{(q^{1/2}-q^{-1/2})J_d(K)}{q^{d/2}-q^{-d/2}}\right|_{q=e^\hbar} = \sum_{j,m\geq 0} a_{jm}(K)d^j\hbar^m,$$

"below diagonal" coefficients vanish, $a_{jm}(K) = 0$ if $j > m$, and "on diagonal" coefficients give the inverse of the Alexander polynomial: $\left(\sum_{m=0}^\infty a_{mm}(K)\hbar^m\right)\cdot\omega(K)(e^\hbar) = 1$.



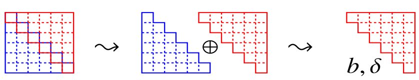
"Above diagonal" we have Rozansky's Theorem [Ro3, (1.2)]:

$$J_d(K)(q) = \frac{q^d-q^{-d}}{(q-q^{-1})\omega(K)(q^d)}\left(1+\sum_{k=1}^\infty \frac{(q-1)^k\rho_k(K)(q^d)}{\omega^{2k}(K)(q^d)}\right).$$

Solvable Approximation. A quantized universal enveloping algebra (aka "quantum group") is an ∞-dimensional inverse limit.

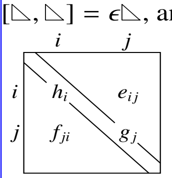
untrimmed	→	halfway-trimmed	→	almost fully-trimmed
↕		↕		↕
too hard		"solvable approximation" (a parameter is hidden)		"finite-type"

Recomposing gl_n. Half is enough! $gl_n \oplus \mathfrak{a}_n = \mathcal{D}(◹, b, \delta)$:



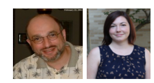
$b(◹) = b\colon ◹\otimes◹ \to ◹$
$b(◺) \rightsquigarrow \delta\colon ◹ \to ◹\otimes◹$

Now define $gl_n^\epsilon := \mathcal{D}(◹, b, \epsilon\delta)$. Schematically, this is $[◹,◹] = ◹$, $[◺,◺] = \epsilon◺$, and $[◹,◺] = ◺ + \epsilon◹$. In detail, it is

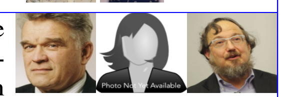


$[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{li}e_{kj}$ $[f_{ij}, f_{kl}] = \epsilon\delta_{jk}f_{il} - \epsilon\delta_{li}f_{kj}$
$[e_{ij}, f_{kl}] = \delta_{jk}(\epsilon\delta_{j<k}e_{il} + \delta_{il}(h_i + \epsilon g_i)/2 + \delta_{i>l}f_{il})$
$\quad -\delta_{li}(\epsilon\delta_{k<j}e_{kj} + \delta_{kj}(h_j + \epsilon g_j)/2 + \delta_{k>j}f_{kj})$
$[g_i, e_{jk}] = (\delta_{ij} - \delta_{ik})e_{jk}$ $[h_i, e_{jk}] = \epsilon(\delta_{ij} - \delta_{ik})e_{jk}$
$[g_i, f_{jk}] = (\delta_{ij} - \delta_{ik})f_{jk}$ $[h_i, f_{jk}] = \epsilon(\delta_{ij} - \delta_{ik})f_{jk}$

Prior art. Some amazing computations by Rozansky and Overbay in [Ro2, Ro3] and in [Ov].


Rozansky, Overbay

Faddeev's Formula (In as much as we can tell, first appeared w/o proof in Faddeev [Fa], rediscovered and proven in Quesne [Qu], and again with easier proof, in Zagier [Za]). With $[n]_q := \frac{q^n-1}{q-1}$, with $[n]_q! := [1]_q[2]_q\cdots[n]_q$ and with $\mathbb{e}_q^x := \sum_{n\geq 0}\frac{x^n}{[n]_q!}$, we have


Faddeev Quesne Zagier

$$\log \mathbb{e}_q^x = \sum_{k\geq 1}\frac{(1-q)^k x^k}{k(1-q^k)} = x + \frac{(1-q)^2x^2}{2(1-q^2)} + \ldots.$$

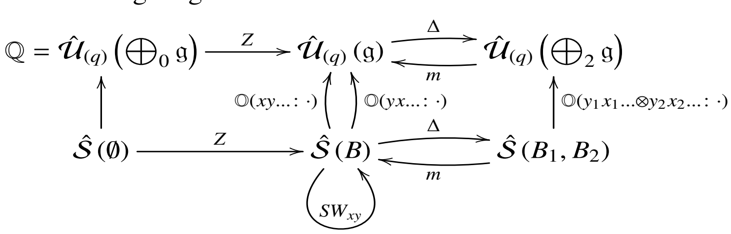
Proof. We have that $\mathbb{e}_q^x = \frac{\mathbb{e}_q^{qx}-\mathbb{e}_q^x}{qx-x}$ ("the q-derivative of $\mathbb{e}_q^x$ is itself"), and hence $\mathbb{e}_q^{qx} = (1+(1-q)x)\mathbb{e}_q^x$, and

$$\log \mathbb{e}_q^{qx} = \log(1+(1-q)x) + \log \mathbb{e}_q^x.$$

Writing $\log \mathbb{e}_q^x = \sum_{k\geq 1}a_kx^k$ and comparing powers of x, we get $q^ka_k = -(1-q)^k/k + a_k$, or $a_k = \frac{(1-q)^k}{k(1-q^k)}$. □

Solvable Approximation (2). At $\epsilon = 1$ and modulo $h = g$, the above is just gl_n. By rescaling at $\epsilon \neq 0$, gl_n^ϵ is independent of ϵ. We let gl_n^k be gl_n^ϵ regarded as an algebra over $\mathbb{Q}[\epsilon]/\epsilon^{k+1} = 0$. It is the "$k$-smidgen solvable approximation" of gl_n!
Recall that $\mathfrak{g}$ is "solvable" if iterated commutators in it ultimately vanish: $\mathfrak{g}_2 := [\mathfrak{g},\mathfrak{g}]$, $\mathfrak{g}_3 := [\mathfrak{g}_2,\mathfrak{g}_2]$, $\ldots$, $\mathfrak{g}_d = 0$. Equivalently, if it is a subalgebra of some large-size ◹ algebra.
Note. This whole process makes sense for arbitrary semi-simple Lie algebras.

***$\mathcal{GDO}$*-Categories.** Given $\mathfrak{g}$ with basis $B = \{x, y, \ldots\}$, consider the following diagram:



Hence Z, SW_{xy}, m, Δ, (and likewise S and θ) are morphisms in the *completion* of the monoidal category $\mathcal{F}$ whose objects are finite sets B and whose morphisms are $\text{mor}_\mathcal{F}(B, B') := \text{Hom}_\mathbb{Q}(\mathcal{S}(B) \to \mathcal{S}(B')) = \mathcal{S}(B^*, B')$ (by convention, $x^* = \xi$, $y^* = \eta$, etc.). Ergo we need to *consolidate* (at least parts of) said completion.

Aside. "Consolidate" means "give a finite name to an infinite object, and figure out how to sufficiently manipulate such finite names". E.g., solving $f'' = -f$ we encounter and set $\sum\frac{(-1)^kx^{2k}}{(2k)!} \rightsquigarrow \cos x$, $\sum\frac{(-1)^kx^{2k+1}}{(2k+1)!} \rightsquigarrow \sin x$, and then $\cos^2 x + \sin^2 x = 1$ and $\sin(x+y) = \sin x\cos y + \cos x \sin y$.

The Composition Law. If

$$\mathcal{S}(B_0) \xrightarrow[{}^t f\in\mathbb{Q}[\![\zeta_{0i}, z_{1j}]\!]]{f} \mathcal{S}(B_1) \xrightarrow[{}^t g\in\mathbb{Q}[\![\zeta_{1j}, z_{2k}]\!]]{g} \mathcal{S}(B_2)$$

then ${}^t(f/\!\!/g) = {}^t(g\circ f) = \left(g|_{\zeta_{1j}\to\partial_{z_{1j}}} f\right)_{z_{1j}=0}$.

Examples.

1. The 1-variable identity map $I\colon \mathcal{S}(z) \to \mathcal{S}(z)$ is given by ${}^tI_1 = \mathbb{e}^{z\zeta}$ and the n-variable one by ${}^tI_n = \mathbb{e}^{z_1\zeta_1+\cdots+z_n\zeta_n}$.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Matemale-1804/