$$
\begin{aligned}
dm_{i,j\to k} \to\ & a_k\,(\alpha_i+\alpha_j) + b_k\,(\beta_i+\beta_j) + y_k\,\eta_i + \frac{y_k\,\eta_j}{\mathcal{A}_i} + \frac{x_k\,\xi_i}{\mathcal{A}_j} + \eta_j\,\xi_i - \\
& B_k\,\eta_j\,\xi_i + \frac{1}{4\,\mathcal{A}_i\,\mathcal{A}_j}\in \Big(2\,y_k\,\eta_j\,\big(2\,x_k\,\xi_i + \mathcal{A}_j\,(-2\,\beta_i + (1-3\,B_k)\,\eta_j\,\xi_i)\big) + \\
& \quad \mathcal{A}_i\,\xi_i\,\Big(x_k\,(-4\,\beta_j + 2\,(1-3\,B_k)\,\eta_j\,\xi_i) + \\
& \qquad \mathcal{A}_j\,\eta_j\,\big(4\,a_k\,B_k + (1-4\,B_k+3\,B_k^2)\,\eta_j\,\xi_i\big)\Big)\Big) + x_k\,\xi_j
\end{aligned}
$$

$$
\begin{aligned}
d\triangle_{i\to j,k} \to\ & a_j\,\alpha_i + a_k\,\alpha_i + b_j\,\beta_i + b_k\,\beta_i + y_j\,\eta_i + B_j\,y_k\,\eta_i + \\
& x_j\,\xi_i + x_k\,\xi_i + \frac{1}{2}\in\big(B_j\,y_j\,y_k\,\eta_i^2 + x_k\,\xi_i\,(-2\,a_j + x_j\,\xi_i)\big)
\end{aligned}
$$

$$
\begin{aligned}
dS_i \to\ & -a_i\,\alpha_i - b_i\,\beta_i - \frac{\mathcal{A}_i\,(y_i\,\eta_i + (-\eta_i + B_i\,(x_i+\eta_i))\,\xi_i)}{B_i} - \\
& \frac{1}{4\,B_i^2}\in \mathcal{A}_i\,\Big(\mathcal{A}_i\,\eta_i^2\,(2\,y_i^2 - 6\,y_i\,\xi_i + 3\,\xi_i^2) + B_i^2\,\xi_i\,\big(4\,a_i\,x_i + 2\,x_i^2\,\mathcal{A}_i\,\xi_i + \\
& \qquad 2\,x_i\,(2\,\beta_i + \mathcal{A}_i\,\eta_i\,\xi_i) + \eta_i\,(-4 + 4\,\beta_i + \mathcal{A}_i\,\eta_i\,\xi_i)\big) + \\
& \quad 2\,B_i\,\eta_i\,\big(y_i\,(-2 + 2\,\beta_i + 2\,x_i\,\mathcal{A}_i\,\xi_i + \mathcal{A}_i\,\eta_i\,\xi_i) - \\
& \qquad \xi_i\,(-2 + 2\,a_i + 2\,\beta_i + 3\,x_i\,\mathcal{A}_i\,\xi_i + 2\,\mathcal{A}_i\,\eta_i\,\xi_i)\big)\Big)
\end{aligned}
$$

$$R_{i,j} \to a_j\,b_i + x_j\,y_i - \frac{1}{4}\in x_j^2\,y_i^2$$

$$P_{i,j} \to \alpha_j\,\beta_i + \eta_i\,\xi_j + \frac{1}{4}\in \eta_i^2\,\xi_j^2$$

$$
\begin{aligned}
\mathbb{E}_{\{\}\to\{1\}}\Big[0,\ 0,\ & \frac{B}{1-B+B^2} + \\
& \frac{B\,\big(-B + 2\,B^2 + 2\,B^4 + a\,(-1 + B - B^3 + B^4) - 2\,x\,y - B^3\,(3 + 2\,x\,y)\big)\in}{(1-B+B^2)^3} + \\
& \frac{1}{2\,(1-B+B^2)^5} \\
& B\,\Big(4\,B^8 + a^2\,(1-B+B^2)^2\,(1 + B - 6\,B^2 + B^3 + B^4) + 6\,B^5\,x^2\,y^2 + \\
& \quad 2\,x\,y\,(-2 + 3\,x\,y) - B^7\,(11 + 4\,x\,y) - 2\,B^2\,(1 + 6\,x^2\,y^2) - \\
& \quad 2\,B^4\,(1 - 2\,x\,y + 6\,x^2\,y^2) + B\,(1 + 8\,x\,y + 6\,x^2\,y^2) + \\
& \quad B^6\,(6 + 8\,x\,y + 6\,x^2\,y^2) + B^3\,(4 + 4\,x\,y + 30\,x^2\,y^2) + \\
& \quad 2\,a\,(1-B+B^2)\,\big(2\,B^6 + 2\,x\,y + 8\,B^3\,(1 + x\,y) - 5\,B^2\,(1 + 2\,x\,y) - \\
& \qquad 2\,B^5\,(1 + 2\,x\,y) - B^4\,(7 + 2\,x\,y) + B\,(2 + 4\,x\,y)\big)\Big)\in^2 + O[\in]^3\Big]
\end{aligned}
$$

A Quantum Algebra Example.

ωεβ/qa

Proto-Proposition†0 (with Jesse Frohlich and Roland van der Veen, near [Ma, Proposition 1.7.3]). Let H be a finite dimensional Hopf algebra and let $U = H^{*cop} \otimes H$ be its Drinfel'd double, with R-matrix $R \in H^* \otimes H \subset U \otimes U$. Write $R^{\dagger 1} = \sum \rho_a \otimes r_a$, and let $\langle \cdot \mid \cdot \rangle \colon H^* \otimes H \to \mathbb{F}$ be the duality pairing. Then the functional $\int \in U^*$ defined by

$$\int \phi \otimes x := \sum \langle \phi \rho_a^{\dagger 2} \mid x r_a^{\dagger 3} \rangle$$

is a right†4 integral in U^*. (Meaning $\Delta^i_{jk} /\!/ \int_j = \int_i /\!/ \epsilon_k$ in $\mathrm{Hom}(U^{\otimes\{i\}} \to U^{\otimes\{k\}})$).

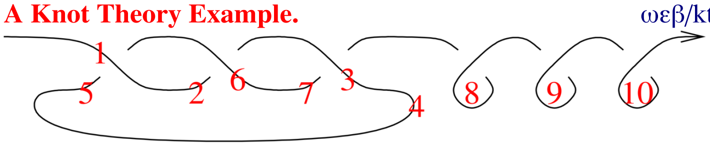
†0 A "proto-proposition" is something that will become a proposition once you figure out the correct statement. †1 Or did we want it to be $R/\!/S_1^2$? Or $R/\!/S_2^2$? †2 Or is it $\rho_a\phi$? †3 Or is it $r_a x$? †4 Or maybe "left"?

```
inp = E_{{}→{1}}[3 a_1 b_1, 5 x_1 y_1, 1] // dm_{i,1→i};
Table[
  HL@TrueQ[
    (inp // (sΥ_{i→1,1,2,2} RR) // BM // AM // P_{1,2} ) dϵ_j ≡
     (inp // ΔΔ // (sΥ_{i→1,1,2,2} RR) // BM // AM // P_{1,2})],
  {ΔΔ, {dΔ_{i→i,j}, dΔ_{i→j,i}}}, {AM, {dm_{2,4→2}, dm_{4,2→2}}},
  {BM, {dm_{1,3→1}, dm_{3,1→1}}},
  {RR, {R_{3,4}, R_{3,4} // dS_3 // dS_3, R_{3,4} // dS_4 // dS_4}}
 ] // MatrixForm
```

$$
\begin{pmatrix}
\begin{pmatrix} \text{False} & \text{False} & \text{False} \\ \text{False} & \text{False} & \text{False} \end{pmatrix} & \begin{pmatrix} \text{False} & \text{False} & \text{True} \\ \text{False} & \text{False} & \text{False} \end{pmatrix} \\
\begin{pmatrix} \text{False} & \text{False} & \text{False} \\ \text{False} & \text{False} & \text{True} \end{pmatrix} & \begin{pmatrix} \text{False} & \text{False} & \text{False} \\ \text{False} & \text{False} & \text{False} \end{pmatrix}
\end{pmatrix}
$$

A Knot Theory Example.

ωεβ/kt




```
$k = 2;
Simplify[
  R_{1,5} R_{6,2} R_{3,7} C̄_4 Kink‾_8 Kink‾_9 Kink‾_10 // dm_{1,2→1} // dm_{1,3→1} //
          dm_{1,4→1} // dm_{1,5→1} // dm_{1,6→1} // dm_{1,7→1} // dm_{1,8→1} //
     dm_{1,9→1} // dm_{1,10→1}] /. v_1 :→ v
```

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Video and more: http://www.math.toronto.edu/~drorbn/Talks/CRM-1907, http://www.math.toronto.edu/~drorbn/Talks/UCLA-191101.