

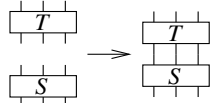


# Geography vs. Identity

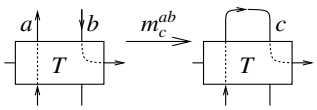
Thanks for inviting me to the *Topology* session!

**Abstract.** Which is better, an emphasis on where things happen or on who are the participants? I can't tell; there are advantages and disadvantages either way. Yet much of quantum topology seems to be heavily and unfairly biased in favour of geography.

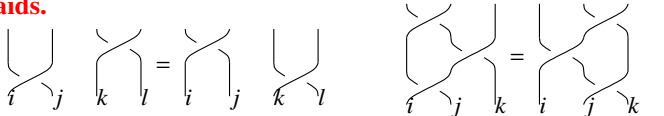
**Geographers** care for placement; for them, braids and tangles have ends at some distinguished points, hence they form categories whose objects are the placements of these points. For them, the basic operation is a binary "stacking of tangles". They are lead to monoidal categories, braided monoidal categories, representation theory, and much or most of we call "quantum topology".



**Identifiers** believe that strand identity persists even if one crosses or is being crossed. The key operation is a unary stitching operation  $m_c^{ab}$ , and one is lead to study meta-monoids, meta-Hopf-algebras, etc. See ωεβ/reg, ωεβ/kbh.



## Braids.



Geography:

$$GB := \langle \gamma_i \rangle \left( \begin{array}{l} \gamma_i \gamma_k = \gamma_k \gamma_i \text{ when } |i - k| > 1 \\ \gamma_i \gamma_{i+1} \gamma_i = \gamma_{i+1} \gamma_i \gamma_{i+1} \end{array} \right) = B.$$

Identity:

(captures quantum algebra!)

$$IB := \langle \sigma_{ij} \rangle \left( \begin{array}{l} \sigma_{ij} \sigma_{kl} = \sigma_{kl} \sigma_{ij} \text{ when } \{|i, j, k, l|\} = 4 \\ \sigma_{ij} \sigma_{ik} \sigma_{jk} = \sigma_{jk} \sigma_{ik} \sigma_{ij} \text{ when } \{|i, j, k|\} = 3 \end{array} \right) = PB.$$

**Theorem.** Let  $S = \{\tau\}$  be the symmetric group. Then  $vB$  is both

$$PB \rtimes S \cong B * S \left( \begin{array}{l} \gamma_i \tau = \tau \gamma_j \text{ when } \tau i = j, \tau(i+1) = (j+1) \end{array} \right)$$

(and so  $PB$  is "bigger" than  $B$ , and hence quantum algebra doesn't see topology very well).

**Proof.** Going left,  $\gamma_i \mapsto \sigma_{i,i+1}(i \ i+1)$ . Going right, if  $i < j$  map  $\sigma_{ij} \mapsto (j-1 \ j-2 \ \dots \ i) \gamma_{j-1}(i \ i+1 \ \dots \ j)$  and if  $i > j$  use  $\sigma_{ij} \mapsto (j \ j+1 \ \dots \ i) \gamma_j(i \ i-1 \ \dots \ j+1)$ .



## The Burau Representation of PB\_n acts on R^n :=

$\mathbb{Z}[t^{\pm 1}]^n = R\langle v_1, \dots, v_n \rangle$  by

$$\sigma_{ij} v_k = v_k + \delta_{kj}(t-1)(v_j - v_i).$$

$\delta /: \delta_{i,j} := \text{If}[i = j, 1, 0];$

ωεβ/code



Werner Burau

$B_{i,j}[\underline{\varepsilon}] := \mathcal{E} / . v_{k-} \mapsto v_k + \delta_{R,j} (t-1) (v_j - v_i) // \text{Expand}$

(bas3 = {v1, v2, v3}) // B1,2

{v1, v1 - t v1 + t v2, v3}

bas3 // B1,2 // B1,3 // B2,3

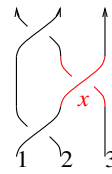
{v1, v1 - t v1 + t v2, v1 - t v1 + t v2 - t^2 v2 + t^2 v3}

bas3 // B2,3 // B1,3 // B1,2

{v1, v1 - t v1 + t v2, v1 - t v1 + t v2 - t^2 v2 + t^2 v3}

$S_n$  acts on  $R^n$  by permuting the  $v_i$  so the Burau representation extends to  $vB_n$  and restricts to  $B_n$ .

With this,  $\gamma_i$  maps  $v_i \mapsto v_{i+1}, v_{i+1} \mapsto t v_i + (1-t) v_{i+1}$ , and otherwise  $v_k \mapsto v_k$ .



## Geography view:

$$\gamma_1 = \text{strand 1 crosses over strand 3} \quad \gamma_2 = \text{strand 2 crosses over strand 3} \quad \gamma_3 = \text{strand 1 crosses over strand 2} \dots$$

so  $x$  is  $\gamma_2$ .

## Identity view:

At  $x$  strand 1 crosses strand 3, so  $x$  is  $\sigma_{13}$ .



**The Gold Standard** is set by the "T-calculus" Alexander formulas (ωεβ/mac). An  $S$ -component tangle  $T$  has

$$\Gamma(T) \in R_S \times M_{S \times S}(R_S) = \left\{ \frac{\omega}{S} \middle| \frac{S}{A} \right\} \text{ with } R_S := \mathbb{Z}\langle T_a : a \in S \rangle:$$

$$(a \overset{*}{\leftarrow} b, b \overset{*}{\leftarrow} a) \rightarrow \frac{1}{a} \begin{array}{c|c} a & b \\ \hline 1 & 1 - T_a^{\pm 1} \end{array} \quad T_1 \sqcup T_2 \rightarrow \frac{\omega_1 \omega_2}{S_1 \ S_2} \begin{array}{c|c} S_1 & S_2 \\ \hline A_1 & 0 \\ 0 & A_2 \end{array}$$

$$\begin{array}{c|c|c} \omega & a & b & S \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi \end{array} \xrightarrow{m_c^{ab}} \begin{array}{c|c} (1-\beta)\omega & c & S \\ \hline c & \gamma + \frac{\alpha\delta}{1-\beta} & \epsilon + \frac{\delta\theta}{1-\beta} \\ S & \phi + \frac{\alpha\psi}{1-\beta} & \Xi + \frac{\psi\theta}{1-\beta} \end{array}$$

## The Gassner Representation of PB\_n acts on V =

$R^n := \mathbb{Z}[t^{\pm 1}, \dots, t_n^{\pm 1}]^n = R\langle v_1, \dots, v_n \rangle$  by

$$\sigma_{ij} v_k = v_k + \delta_{kj}(t_i - 1)(v_j - v_i).$$



Betty Jane Gassner deserves to be more famous

$G_{i,j}[\underline{\varepsilon}] := \mathcal{E} / . v_{k-} \mapsto v_k + \delta_{R,j} (t_i - 1) (v_j - v_i) // \text{Expand}$

(bas3 // G1,2 // G1,3 // G2,3) = (bas3 // G2,3 // G1,3 // G1,2)

True

$S_n$  acts on  $R^n$  by permuting the  $v_i$  and the  $t_i$ , so the Gassner representation extends to  $vB_n$  and then restricts to  $B_n$  as a  $\mathbb{Z}$ -linear  $\infty$ -dimensional representation. It then descends to  $PB_n$  as a finite-rank  $R$ -linear representation, with lengthy non-local formulas.

**Geographers:** Gassner is an obscure partial extension of Burau.

**Identifiers:** Burau is a trivial silly reduction of Gassner.

## The Turbo-Gassner Representation. With the same

$R$  and  $V$ ,  $TG$  acts on  $V \oplus (R^n \otimes V) \oplus (S^2 V \otimes V^*) =$

$R\langle v_k, v_{lk}, u_i u_j w_k \rangle$  by

$$TG_{i,j}[\underline{\varepsilon}] := \mathcal{E} / . \left\{ \begin{array}{l} v_{k-} \mapsto v_k + \delta_{R,j} ((t_i - 1) (v_j - v_i) + v_{i,j} - v_{i,i}) + \delta_{R,i} (u_j - u_i) u_i w_j, \\ v_{l-,k} \mapsto v_{l-,k} + (t_i - 1) \times (\delta_{R,j} (v_{l-,j} - v_{l-,i}) + (\delta_{l-,i} - \delta_{l-,j} t_i^{-1} t_j) (u_k + \delta_{R,j} (t_i - 1) (u_j - u_i)) u_i w_j), \\ u_{i-} \mapsto u_i + \delta_{R,j} (t_i - 1) (u_j - u_i), \\ w_{i-} \mapsto w_i + (\delta_{R,j} - \delta_{R,i}) (t_i^{-1} - 1) w_j // \text{Expand} \end{array} \right.$$



With Roland van der Veen

Gassner motifs  
Adjoint-Gassner

$$\text{bas3} = \{v_1, v_2, v_3, v_{1,1}, v_{1,2}, v_{1,3}, v_{2,1}, v_{2,2}, v_{2,3}, v_{3,1}, v_{3,2}, v_{3,3}, u_1^2 w_1, u_1^2 w_2, u_1^2 w_3, u_1 u_2 w_1, u_1 u_2 w_2, u_1 u_2 w_3, u_1 u_3 w_1, u_1 u_3 w_2, u_1 u_3 w_3, u_2^2 w_1, u_2^2 w_2, u_2^2 w_3, u_2 u_3 w_1, u_2 u_3 w_2, u_2 u_3 w_3, u_3^2 w_1, u_3^2 w_2, u_3^2 w_3\};$$

$$(\text{bas3} // TG_{1,2} // TG_{1,3} // TG_{2,3}) = (\text{bas3} // TG_{2,3} // TG_{1,3} // TG_{1,2})$$

True

Like Gassner,  $TG$  is also a representation of  $PB_n$ .

I have no idea where it belongs!

My talk tomorrow, at the *chord diagrams everywhere* session:

