

The PBW Principle Lots of algebras are isomorphic as vector spaces to polynomial algebras. So we want to understand arbitrary linear maps between polynomial algebras.

Convention. For a finite set A , let $z_A := \{z_i\}_{i \in A}$ and let $\zeta_A := \{\zeta_i^* = \zeta_i\}_{i \in A}$.

The Generating Series \mathcal{G} : $\text{Hom}(\mathbb{Q}[z_A] \rightarrow \mathbb{Q}[z_B]) \rightarrow \mathbb{Q}[\zeta_A, z_B]$.

Claim. $L \in \text{Hom}(\mathbb{Q}[z_A] \rightarrow \mathbb{Q}[z_B]) \xrightarrow{\mathcal{G}} \mathbb{Q}[\zeta_A, z_B] \ni \mathcal{L}$ via

$$\mathcal{G}(L) := \sum_{n \in \mathbb{N}^A} \frac{\zeta_A^n}{n!} L(z_A^n) = L\left(\prod_{a \in A} \zeta_a z_a\right) = \mathcal{L} = \text{greek } \mathcal{L}_{\text{latin}}$$

$$\mathcal{G}^{-1}(\mathcal{L})(p) = \left(p|_{z_a \rightarrow \partial_{z_a} \mathcal{L}} \right)_{\zeta_a=0} \quad \text{for } p \in \mathbb{Q}[z_A].$$

Claim. If $L \in \text{Hom}(\mathbb{Q}[z_A] \rightarrow \mathbb{Q}[z_B])$, $M \in \text{Hom}(\mathbb{Q}[z_B] \rightarrow \mathbb{Q}[z_C])$, then $\mathcal{G}(L \circ M) = (\mathcal{G}(L))|_{z_b \rightarrow \partial_{z_b} \mathcal{G}(M)}_{\zeta_b=0}$.

Examples. • $\mathcal{G}(\text{id}: \mathbb{Q}[p, x] \rightarrow \mathbb{Q}[p, x]) = e^{\pi p + \xi x}$.

• Consider $R_{ij} \in (\mathfrak{h}_i \otimes \mathfrak{h}_j)[[t]] \cong \text{Hom}(\mathbb{Q}[\] \rightarrow \mathbb{Q}[p_i, x_i, p_j, x_j])[[t]]$.

Then $\mathcal{G}(R_{ij}) = e^{(e^t - 1)(p_i - p_j)x_j} = e^{(T - 1)(p_i - p_j)x_j}$.

Heisenberg Algebras. Let $\mathfrak{h} = A\langle p, x \rangle / ([p, x] = 1)$, let $\mathcal{O}_i: \mathbb{Q}[p_i, x_i] \rightarrow \mathfrak{h}_i$ is the “ p before x ” PBW normal ordering map and let hm_k^{ij} be the composition

$$\mathbb{Q}[p_i, x_i, p_j, x_j] \xrightarrow{\mathcal{O}_i \otimes \mathcal{O}_j} \mathfrak{h}_i \otimes \mathfrak{h}_j \xrightarrow{m_k^{ij}} \mathfrak{h}_k \xrightarrow{\mathcal{O}_k^{-1}} \mathbb{Q}[p_k, x_k].$$

Then $\mathcal{G}(hm_k^{ij}) = e^{-\xi_i \pi_j + (\pi_i + \pi_j)p_k + (\xi_i + \xi_j)x_k}$.

Proof. Recall the “Weyl CCR” $e^{\xi x} e^{\pi p} = e^{-\xi \pi} e^{\pi p} e^{\xi x}$, and find

$$\begin{aligned} \mathcal{G}(hm_k^{ij}) &= e^{\pi_i p_i + \xi_i x_i + \pi_j p_j + \xi_j x_j} // \mathcal{O}_i \otimes \mathcal{O}_j // m_k^{ij} // \mathcal{O}_k^{-1} \\ &= e^{\pi_i p_i} e^{\xi_i x_i} e^{\pi_j p_j} e^{\xi_j x_j} // m_k^{ij} // \mathcal{O}_k^{-1} = e^{\pi_i p_k} e^{\xi_i x_k} e^{\pi_j p_k} e^{\xi_j x_k} // \mathcal{O}_k^{-1} \\ &= e^{-\xi_i \pi_j} e^{(\pi_i + \pi_j)p_k} e^{(\xi_i + \xi_j)x_k} // \mathcal{O}_k^{-1} = e^{-\xi_i \pi_j + (\pi_i + \pi_j)p_k + (\xi_i + \xi_j)x_k}. \end{aligned}$$

GDO := The category with objects finite sets and

$$\text{mor}(A \rightarrow B) = \{ \mathcal{L} = \omega e^{\mathcal{Q}} \} \subset \mathbb{Q}[\zeta_A, z_B],$$

where: • ω is a scalar. • \mathcal{Q} is a “small” quadratic in $\zeta_A \cup z_B$.

• Compositions: $\mathcal{L} // \mathcal{M} := (\mathcal{L})|_{z_i \rightarrow \partial_{z_i} \mathcal{M}}_{\zeta_i=0}$.

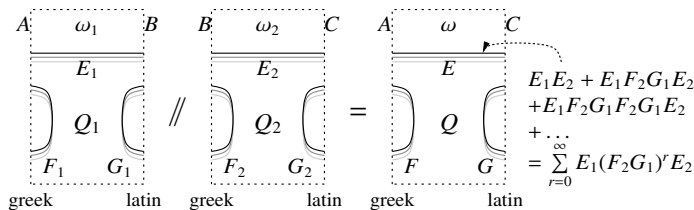
Compositions. In $\text{mor}(A \rightarrow B)$,

$$Q = \sum_{i \in A, j \in B} E_{ij} \zeta_i z_j + \frac{1}{2} \sum_{i, j \in A} F_{ij} \zeta_i \zeta_j + \frac{1}{2} \sum_{i, j \in B} G_{ij} z_i z_j,$$



R. Feynman

and so (remember, $e^x = 1 + x + xx/2 + xxx/6 + \dots$)



where • $E = E_1(I - F_2 G_1)^{-1} E_2$ • $F = F_1 + E_1 F_2 (I - G_1 F_2)^{-1} E_1^T$ • $G = G_2 + E_2^T G_1 (I - F_2 G_1)^{-1} E_2$ • $\omega = \omega_1 \omega_2 \det(I - F_2 G_1)^{-1/2}$

Proof of Claim in Example 2. Let $\Phi_1 := e^{t(p_i - p_j)x_j}$ and $\Phi_2 := \mathcal{O}_{p_j x_j}(e^{(e^t - 1)(p_i - p_j)x_j}) =: \mathcal{O}(\Psi)$. We show that $\Phi_1 = \Phi_2$ in $(\mathfrak{h}_i \otimes \mathfrak{h}_j)[[t]]$ by showing that both solve the ODE $\partial_t \Phi = (p_i - p_j)x_j \Phi$ with $\Phi|_{t=0} = 1$. For Φ_1 this is trivial. $\Phi_2|_{t=0} = 1$ is trivial, and

$$\partial_t \Phi_2 = \mathcal{O}(\partial_t \Psi) = \mathcal{O}(e^t (p_i - p_j)x_j \Psi)$$

$$(p_i - p_j)x_j \Phi_2 = (p_i - p_j)x_j \mathcal{O}(\Psi) = (p_i - p_j)\mathcal{O}(x_j \Psi - \partial_{p_j} \Psi)$$

$$= \mathcal{O}((p_i - p_j)(x_j \Psi + (e^t - 1)x_j \Psi)) = \mathcal{O}(e^t (p_i - p_j)x_j \Psi) \quad \square$$

Implementation.

Without, don't trust!

CF = ExpandNumerator*ExpandDenominator*PowerExpand*Factor;

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EA1 -> B1 [omega1, Q1_] EA2 -> B2 [omega2, Q2_] ^:= EA1UA2->B1UB2 [omega1 omega2, Q1 + Q2]
(EA1 -> B1 [omega1, Q1_] // EA2 -> B2 [omega2, Q2_] ) /; (B1* == A2) :=
Module[{i, j, E1, F1, G1, E2, F2, G2, I, M = Table},
  I = IdentityMatrix@Length@B1;
  E1 = M[theta_i, j, Q1, {i, A1}, {j, B1}]; E2 = M[theta_i, j, Q2, {i, A2}, {j, B2}];
  F1 = M[theta_i, j, Q1, {i, A1}, {j, A1}]; F2 = M[theta_i, j, Q2, {i, A2}, {j, A2}];
  G1 = M[theta_i, j, Q1, {i, B1}, {j, B1}]; G2 = M[theta_i, j, Q2, {i, B2}, {j, B2}];
  EA1 -> B2 [CF [omega1 omega2 Det[I - F2.G1]^(1/2)], CF@Plus[
    If[A1 == {} || B2 == {}, 0, A1.E1.Inverse[I - F2.G1].E2.B2],
    If[A1 == {}, 0, 1/2.A1.(F1 + E1.F2.Inverse[I - G1.F2].E1^T).A1],
    If[B2 == {}, 0, 1/2.B2.(G2 + E2^T.G1.Inverse[I - F2.G1].E2).B2]]]]]
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A \ B := Complement[A, B];
(EA1 -> B1 [omega1, Q1_] // EA2 -> B2 [omega2, Q2_] ) /; (B1* != A2) :=
EA1U(A2 \ B1*) -> B1UA2* [omega1, Q1 + Sum[epsilon^* xi, {xi, A2 \ B1*}]] //
EA1UA2 -> B2U(B1 \ A2*) [omega2, Q2 + Sum[z^* z, {z, B1 \ A2*}]]
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{p^*, x^*, pi^*, xi^*} = {pi, xi, p, x}; (u_i^*)^* := (u^*)_i;
L_LiSt^* := #* & /& L;
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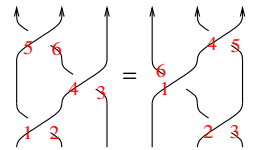
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R_i, j_ := E_{i -> {p_i, x_i, p_j, x_j}} [T^-1/2, (1 - T) p_j x_j + (T - 1) p_i x_j];
R_bar_i, j_ := E_{i -> {p_i, x_i, p_j, x_j}} [T^1/2, (1 - T^-1) p_j x_j + (T^-1 - 1) p_i x_j];
C_i_ := E_{i -> {p_i, x_i}} [T^-1/2, 0];
C_bar_i_ := E_{i -> {p_i, x_i}} [T^1/2, 0];
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hm_i, j_ -> k_ := E_{pi_i, xi_i, pi_j, xi_j -> {p_k, x_k}} [1, -xi_i pi_j + (pi_i + pi_j) p_k + (xi_i + xi_j) x_k]
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E_{i -> vs_ [omega_i, Q_i]_h := Module[{ps, xs, M},
  ps = Cases[vs, p_]; xs = Cases[vs, x_];
  M = Table[omega_i, 1 + Length@ps, 1 + Length@xs];
  M[[2 ;;, 2 ;;]] = Table[CF[theta_i, j, Q_i], {i, ps}, {j, xs}];
  M[[2 ;;, 1]] = ps; M[[1, 2 ;;]] = xs;
  MatrixForm[M]_h]
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Proof of Reidemeister 3.

$$(R_{1,2} R_{4,3} R_{5,6} // hm_{1,4 \rightarrow 1} hm_{2,5 \rightarrow 2} hm_{3,6 \rightarrow 3}) == (R_{2,3} R_{1,6} R_{4,5} // hm_{1,4 \rightarrow 1} hm_{2,5 \rightarrow 2} hm_{3,6 \rightarrow 3})$$



True

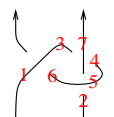
The “First Tangle”.

Factor /@

$$(z = R_{1,6} \bar{C}_3 \bar{R}_{7,4} \bar{R}_{5,2} // hm_{1,3 \rightarrow 1} // hm_{1,4 \rightarrow 1} // hm_{1,5 \rightarrow 1} // hm_{1,6 \rightarrow 1} // hm_{2,7 \rightarrow 2})$$

$$E_{i -> \{p_1, p_2, x_1, x_2\}} \left[\frac{-1 + 2T}{T}, \frac{(-1 + T)(p_1 - p_2)(T x_1 - x_2)}{-1 + 2T} \right]$$

$$z_h = \begin{pmatrix} \frac{-1+2T}{T} & x_1 & x_2 \\ p_1 & \frac{-T-T^2}{-1+2T} & \frac{1-T}{-1+2T} \\ p_2 & \frac{T-T^2}{-1+2T} & \frac{-1-T}{-1+2T} \end{pmatrix}_h$$

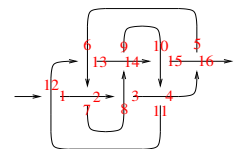


The knot 8₁₇.

$$z = \bar{R}_{12,1} \bar{R}_{27} \bar{R}_{83} \bar{R}_{4,11} \bar{R}_{16,5} \bar{R}_{6,13} \bar{R}_{14,9} \bar{R}_{10,15};$$

Table[z = z // hm_{1k \rightarrow 1}, {k, 2, 16}] // Last

$$E_{i -> \{p_1, x_1\}} \left[\frac{1 - 4T + 8T^2 - 11T^3 + 8T^4 - 4T^5 + T^6}{T^3}, 0 \right]$$



Proof of Theorem 3, (3).

$$\left\{ \gamma 1 = E_{i -> \{p_1, x_1, p_2, x_2, p_3, x_3\}} \left[\omega, \{p_1, p_2, p_3\} \cdot \begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \xi \end{pmatrix} \cdot \{x_1, x_2, x_3\} \right]_h \right\}$$

$$(\gamma 1 // hm_{1,2 \rightarrow \emptyset})_h$$

$$\left\{ \begin{pmatrix} \omega & x_1 & x_2 & x_3 \\ p_1 & \alpha & \beta & \theta \\ p_2 & \gamma & \delta & \epsilon \\ p_3 & \phi & \psi & \xi \end{pmatrix}_h, \begin{pmatrix} \omega + \gamma \omega & x_0 \\ p_0 & \frac{\alpha\beta + \gamma\delta + \theta - \alpha\delta}{1 + \gamma} & \frac{\epsilon - \alpha\epsilon + \theta\gamma\theta}{1 + \gamma} \\ p_3 & \frac{\phi - \delta\phi + \psi + \gamma\psi}{1 + \gamma} & \frac{\xi + \gamma\xi - \epsilon\phi}{1 + \gamma} \end{pmatrix}_h \right\}$$

References.

On $\omega\epsilon\beta = \text{http://drorbn.net/cat20}$