

## 5. Some Problems in Heaven

Unfortunately,  $\dim \mathcal{A}(\mathcal{X}, X) = \dim \Lambda(\mathcal{X}, X) = 4^{|\mathcal{X}|}$  is big. Fortunately, we have the following theorem, a version of one of the main results in Halacheva's thesis, [Ha1, Ha2]:

**Theorem.** Working in  $\Lambda(\mathcal{X} \cup X)$ , if  $w = \omega e^\lambda$  is a balanced Gaussian (namely, a scalar  $\omega$  times the exponential of a quadratic  $\lambda = \sum_{\zeta \in \mathcal{X}, z \in X} \alpha_{\zeta, z} \zeta z$ ), then generically so is  $c_{x, \xi} e^\lambda$ . (This is great news! The space of balanced quadratics is only  $|\mathcal{X}| |X|$ -dimensional!)

**Proof.** Recall that  $c_{x, \xi}: (1, \xi, x, x\xi)w' \mapsto (1, 0, 0, 1)w'$ , write  $\lambda = \mu + \eta x + \xi y + \alpha \xi x$ , and ponder  $e^\lambda =$

$$\dots + \frac{1}{k!} \underbrace{(\mu + \eta x + \xi y + \alpha \xi x)(\mu + \eta x + \xi y + \alpha \xi x) \cdots (\mu + \eta x + \xi y + \alpha \xi x)}_{k \text{ factors}} + \dots$$

Then  $c_{x, \xi} e^\lambda$  has three contributions:

- ▶  $e^\mu$ , from the term proportional to 1 (namely, independent of  $\xi$  and  $x$ ) in  $e^\lambda$
- ▶  $-\alpha e^\mu$ , from the term proportional to  $x\xi$ , where the  $x$  and the  $\xi$  come from the same factor above.
- ▶  $\eta y e^\mu$ , from the term proportional to  $x\xi$ , where the  $x$  and the  $\xi$  come from different factors above.

So  $c_{x, \xi} e^\lambda = e^\mu(1 - \alpha + \eta y) = (1 - \alpha)e^\mu(1 + \eta y/(1 - \alpha)) = (1 - \alpha)e^{\mu + \eta y/(1 - \alpha)}$ .

□

## $\Gamma$ -calculus.

Thus we have an almost-always-defined “ $\Gamma$ -calculus”: a contraction algebra morphism  $\mathcal{T}(\mathcal{X}, X) \rightarrow R \times (\mathcal{X} \otimes_{R/R} X)$  whose behaviour under contractions is given by

$$c_{x, \xi}(\omega, \lambda = \mu + \eta x + \xi y + \alpha \xi x) = ((1 - \alpha)\omega, \mu + \eta y/(1 - \alpha)).$$

( $\Gamma$  is fully defined on pure tangles – tangles without closed components – and hence on long knots).

Multiplying and comparing  $\Gamma$  objects:

```

Γ /: Γ[is1_, os1_, cs1_, ω1_, λ1_] × Γ[is2_, os2_, cs2_, ω2_, λ2_] :=
  Γ[is1 ∪ is2, os1 ∪ os2, Join[cs1, cs2], ω1 ω2, λ1 + λ2]
Γ /: Γ[is1_, os1_, ω1_, λ1_] ≡ Γ[is2_, os2_, ω2_, λ2_] :=
  TrueQ[Sort@is1 == Sort@is2] ∧ (Sort@os1 == Sort@os2) ∧
  Simplify[ω1 == ω2] ∧ CF@λ1 == CF@λ2

```

No rules for linear operations!

The crossings and the point:

```

Γ[Xi,j,k,l[S-, T-]] := Γ[{l, i}, {j, k}, <ξi → S, xj → T, xk → S, ξl → T>,
  T-1/2, CF[{ξi, ξl}. (1 1 - T; 0 0 T). {xj, xk}]];
Γ[X̄i,j,k,l[S-, T-]] := Γ[{i, j}, {k, l}, <ξi → S, ξj → T, xk → S, xl → T>,
  T1/2, CF[{ξi, ξj}. (T-1 0; 1 - T-1 1). {xk, xl}]];
Γ[Xi,j,k,l] := Γ[Xi,j,k,l[τi, τl]];
Γ[X̄i,j,k,l] := Γ[X̄i,j,k,l[τi, τj]];
Γ[Pi,j[T-]] := Γ[{i}, {j}, <ξi → T, xj → T>, 1, ξi xj];
Γ[Pi,j] := Γ[Pi,j[τi]];

```

## 6. An Implementation of $\Gamma$ .

If I didn't implement I wouldn't believe myself.

Written in Mathematica [Wo], available as the notebook Gamma.nb at <http://drorbn.net/mo21/ap>. Code lines are highlighted in grey, demo lines are plain. We start with canonical forms for quadratics with rational function coefficients:

```

CCF[ξ_] := Factor[ξ];
CF[ξ_] := Module[{vs = Union@Cases[ξ, {ξ | x}_, ∞]},
  Total[(CCF[#][2]] (Times @@ vs#[[1]])] & /@ CoefficientRules[ξ, vs]];

```

Contractions:

```

ch,t@Γ[is_, os_, cs_, ω_, λ_] := Module[{α, η, γ, μ},
  α = D[ξt, xh]; μ = λ /. ξt | xh → 0;
  η = D[xh, ξt]; γ = D[ξt, λ /. xh → 0];
  Γ[
    DeleteCases[is, t], DeleteCases[os, h], KeyDrop[cs, {xh, ξt}],
    CCF[(1 - α) ω], CF[μ + η γ / (1 - α)]
  ] /. If[MatchQ[cs[ξt], τl], cs[ξt] → cs[xh], cs[xh] → cs[ξt]];
c@Γ[is_, os_, cs_, ω_, λ_] := Fold[ch2,h2[#1] &, Γ[is, os, cs, ω, λ], is ∩ os]

```

Automatic intelligent contractions:

```

Γ[{γ-T}] := c[γ];
Γ[{γ1-T, γ2-T}] := Module[{γ2},
  γ2 = First@MaximalBy[{γS}, Length[γ1[[1]] ∩ #[[2]]] + Length[γ2[[2]] ∩ #[[1]]] &];
  Γ[Join[{c[γ1 γ2]}, DeleteCases[{γS}, γ2]]];
Γ[os_List] := Γ[Γ /@ os]

```