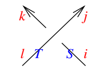
Contractions!

```
c_{x_,y_}[w_Wedge] := Module[{i, j},
  {i} = FirstPosition[w, x, {0}]; {j} = FirstPosition[w, y, {0}];
  { w                                                   (i == 0) ∧ (j == 0)
  { (-1)^(i+j+If[i>j,0,1]) Delete[w, {{i}, {j}}]        (i > 0) ∧ (j > 0)
];
c_{x_,y_}[ℰ_] := ℰ /. w_Wedge :> c_{x,y}[w]
WExp[a∧b + 2 c∧d]
c_{d,c}@WExp[a∧b + 2 c∧d]

Wedge[] + a∧b + 2 c∧d + 2 a∧b∧c∧d
-Wedge[] - a∧b
```



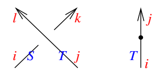
$X_{ijkl}[S,T]$

𝒜[is,os,cs,w] is also a container for the values of the 𝒜-invariant of a tangle. In it, is are the labels of the input strands, os are the labels of the output strands, cs is an assignment of colours (namely, variables) to all the ends $\{\xi_i\}_{i\in \text{is}} \sqcup \{x_j\}_{j\in \text{os}}$, and w is the "payload": an element of $\Lambda(\{\xi_i\}_{i\in \text{is}} \sqcup \{x_j\}_{j\in \text{os}})$.

```
𝒜[X_{i_,j_,k_,l_}[S_, T_]] := 𝒜[{l, i}, {j, k}, <|ξ_i → S, x_j → T, x_k → S, ξ_l → T|>,
  Expand[T^{-1/2} WExp[Expand[{ξ_l, ξ_i}.(1  1-T
                                           0    T ).{x_j, x_k}] /. ξ_a_ x_b_ :> ξ_a∧x_b]]];
𝒜[X_{1,2,3,4}[u, v]]

𝒜[{4, 1}, {2, 3}, <|ξ_1 → u, x_2 → v, x_3 → u, ξ_4 → v|>,
  Wedge[]/√v - (x_2∧ξ_4)/√v - √v x_3∧ξ_1 - (x_3∧ξ_4)/√v + √v x_3∧ξ_4 + √v x_2∧x_3∧ξ_1∧ξ_4]
𝒜[X̄_{i_,j_,k_,l_}] := 𝒜[X̄_{i,j,k,l}[τ_i, τ_j]];
```

The negative crossing and the "point":



$\bar{X}_{ijkl}[S,T]$ $P_{ij}[T]$

```
𝒜[X̄_{i_,j_,k_,l_}[S_, T_]] := 𝒜[{i, j}, {k, l}, <|ξ_i → S, ξ_j → T, x_k → S, x_l → T|>,
  Expand[T^{1/2} WExp[Expand[{ξ_i, ξ_j}.( T^-1      0
                                          1 - T^-1  1 ).{x_k, x_l}] /. ξ_a_ x_b_ :> ξ_a∧x_b]]];
𝒜[X_{i_,j_,k_,l_}] := 𝒜[X_{i,j,k,l}[τ_i, τ_l]];
𝒜[P_{i_,j_}[T_]] := 𝒜[{i}, {j}, <|ξ_i → T, x_j → T|>, WExp[ξ_i∧x_j]];
𝒜[P_{i_,j_}] := 𝒜[P_{i,j}[τ_i]];
```

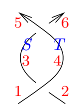
The linear structure on 𝒜's:

```
𝒜 /: α_ × 𝒜[is_, os_, cs_, w_] := 𝒜[is, os, cs, Expand[α w]]
𝒜 /: 𝒜[is1_, os1_, cs1_, w1_] + 𝒜[is2_, os2_, cs2_, w2_] /;
  (Sort@is1 == Sort@is2) ∧ (Sort@os1 == Sort@os2) ∧
   (Sort@Normal@cs1 == Sort@Normal@cs2) := 𝒜[is1, os1, cs1, w1 + w2]
```

Deciding if two 𝒜's are equal:

```
𝒜 /: 𝒜[is1_, os1_, _, w1_] ≡ 𝒜[is2_, os2_, _, w2_] :=
 TrueQ[(Sort@is1 === Sort@is2) ∧ (Sort@os1 === Sort@os2) ∧
   PowerExpand[w1 == w2]]
```

The union operation on 𝒜's (implemented as "multiplication"):




```
𝒜 /: 𝒜[is1_, os1_, cs1_, w1_] × 𝒜[is2_, os2_, cs2_, w2_] :=
 𝒜[is1 ⋃ is2, os1 ⋃ os2, Join[cs1, cs2], WP[w1, w2]]
Short[𝒜[X_{2,4,3,1}[S, T]] × 𝒜[X̄_{3,4,6,5}], 5]

𝒜[{1, 2, 3, 4}, {3, 4, 5, 6},
 <|ξ_2 → S, x_4 → T, x_3 → S, ξ_1 → T, ξ_3 → τ_3, ξ_4 → τ_4, x_6 → τ_3, x_5 → τ_4|>, (√τ_4 Wedge[])/√T -
  (√τ_4 x_3∧ξ_1)/√T + √T √τ_4 x_3∧ξ_1 - √T √τ_4 x_3∧ξ_2 - (√τ_4 x_4∧ξ_1)/√T - (√τ_4 x_5∧ξ_4)/√T -
  (x_6∧ξ_3)/(√T √τ_4) + <<40>> + (√T x_3∧x_5∧x_6∧ξ_1∧ξ_3∧ξ_4)/√τ_4 - (√T x_3∧x_5∧x_6∧ξ_2∧ξ_3∧ξ_4)/√τ_4 -
  (x_4∧x_5∧x_6∧ξ_1∧ξ_3∧ξ_4)/(√T √τ_4) + (√T x_3∧x_4∧x_5∧x_6∧ξ_1∧ξ_2∧ξ_3∧ξ_4)/√τ_4]
```

Contractions of 𝒜-objects:

```
c_{h_,t_}@𝒜[is_, os_, cs_, w_] := 𝒜[
   DeleteCases[is, t], DeleteCases[os, h], KeyDrop[cs, {x_h, ξ_t}], c_{x_h,ξ_t}[w]
  ] /. If[MatchQ[cs[ξ_t], τ_], cs[ξ_t] → cs[x_h], cs[x_h] → cs[ξ_t]];
c_{4,4}[𝒜[X_{2,4,3,1}[S, T]] × 𝒜[X̄_{3,4,6,5}]]

𝒜[{1, 2, 3}, {3, 5, 6}, <|ξ_2 → S, x_3 → S, ξ_1 → T, ξ_3 → τ_3, x_6 → τ_3, x_5 → T|>,
 Wedge[] - x_3∧ξ_1 + T x_3∧ξ_1 - T x_3∧ξ_2 - x_5∧ξ_1 - x_6∧ξ_1 + (x_6∧ξ_1)/T - (x_6∧ξ_3)/T +
  T x_3∧x_5∧ξ_1∧ξ_2 - x_3∧x_6∧ξ_1∧ξ_2 + T x_3∧x_6∧ξ_1∧ξ_2 + x_3∧x_6∧ξ_1∧ξ_3 -
  (x_3∧x_6∧ξ_1∧ξ_3)/T - x_3∧x_6∧ξ_2∧ξ_3 - (x_5∧x_6∧ξ_1∧ξ_3)/T - x_3∧x_5∧x_6∧ξ_1∧ξ_2∧ξ_3]
```

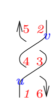
Automatic and intelligent multiple contractions:

```
c@𝒜[is_, os_, cs_, w_] := Fold[c_{#2,#2}[#1] &, 𝒜[is, os, cs, w], is ⋂ os]
𝒜[{A_𝒜}] := c[A];
𝒜[{A1_𝒜, As__𝒜}] := Module[{A2},
  A2 = First@MaximalBy[{As}, Length[A1⟦1⟧ ⋂ #⟦2⟧] + Length[A1⟦2⟧ ⋂ #⟦1⟧] &];
  𝒜[Join[{c[A1 A2]}, DeleteCases[{As}, A2]]]]
𝒜[𝒪s_List] := 𝒜[𝒜 /@ 𝒪s]
c[𝒜[X_{2,4,3,1}[S, T]] × 𝒜[X̄_{3,4,6,5}]]

𝒜[{1, 2}, {5, 6}, <|ξ_2 → S, ξ_1 → T, x_6 → S, x_5 → T|>,
 Wedge[] - x_5∧ξ_1 - x_6∧ξ_2 - x_5∧x_6∧ξ_1∧ξ_2]
𝒜@{𝒜[X_{2,4,3,1}[S, T]], 𝒜[X̄_{3,4,6,5}]}

𝒜[{1, 2}, {5, 6}, <|ξ_2 → S, ξ_1 → T, x_6 → S, x_5 → T|>,
 Wedge[] - x_5∧ξ_1 - x_6∧ξ_2 - x_5∧x_6∧ξ_1∧ξ_2]
```

4. Skein relations and evaluations for 𝒜




```
𝒜@{X̄_{4,1,6,3}[v, u], X̄_{3,2,5,4}}

𝒜[{1, 2}, {5, 6}, <|ξ_2 → v, x_5 → u, ξ_1 → u, x_6 → v|>,
 √u √v Wedge[] - (√u x_5∧ξ_1)/√v + (√u x_5∧ξ_2)/√v - √u √v x_5∧ξ_2 + (√v x_6∧ξ_1)/√u - √u √v x_6∧ξ_1 -
  (√v x_6∧ξ_2)/√u - (√u x_5∧x_6∧ξ_1∧ξ_2)/√v - (√v x_5∧x_6∧ξ_1∧ξ_2)/√u + √u √v x_5∧x_6∧ξ_1∧ξ_2]
```