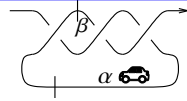
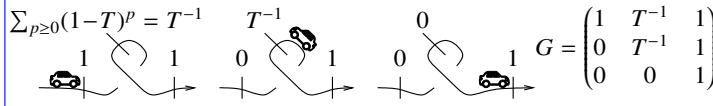


**Theorem.** The Green function  $g_{\alpha\beta}$  is the reading of a traffic counter at  $\beta$ , if car traffic is injected at  $\alpha$  (if  $\alpha = \beta$ , the counter is *after* the injection point).



**Example.**



**Proof.** Near a crossing  $c$  with sign  $s$ , incoming upper edge  $i$  and incoming lower edge  $j$ , both sides satisfy the  $g$ -rules:

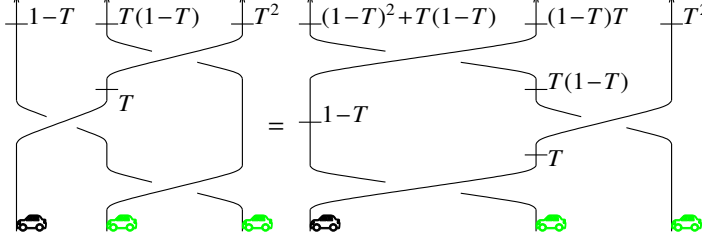
$$g_{i\beta} = \delta_{i\beta} + T^s g_{i+1,\beta} + (1 - T^s) g_{j+1,\beta}, \quad g_{j\beta} = \delta_{j\beta} + g_{j+1,\beta},$$

and always,  $g_{\alpha,2n+1} = 1$ : use common sense and  $AG = I (= GA)$ .

**Bonus.** Near  $c$ , both sides satisfy the further  $g$ -rules:

$$g_{\alpha i} = T^{-s}(g_{\alpha,i+1} - \delta_{\alpha,i+1}), \quad g_{\alpha j} = g_{\alpha,j+1} - (1 - T^s)g_{\alpha i} - \delta_{\alpha,j+1}.$$

**Invariance of  $\rho_1$ .** We start with the hardest, Reidemeister 3:



⇒ Overall traffic patterns are unaffected by Reid3!

⇒ Green's  $g_{\alpha\beta}$  is unchanged by Reid3, provided the cars injection site  $\alpha$  and the traffic counters  $\beta$  are away.

⇒ Only the contribution from the  $R_1$  terms within the Reid3 move matters, and using  $g$ -rules the relevant  $g_{\alpha\beta}$ 's can be pushed outside of the Reid3 area:

$$\delta_{i,j} := \text{If}[i = j, 1, 0];$$

$$\text{gRules}_{s,i,j} :=$$

$$\{ g_{i\beta} \mapsto \delta_{i\beta} + T^s g_{i+1,\beta} + (1 - T^s) g_{j+1,\beta},$$

$$g_{j\beta} \mapsto \delta_{j\beta} + g_{j+1,\beta}, \quad g_{\alpha,i} \mapsto T^{-s} (g_{\alpha,i+1} - \delta_{\alpha,i+1}),$$

$$g_{\alpha,j} \mapsto g_{\alpha,j+1} - (1 - T^s) g_{\alpha i} - \delta_{\alpha,j+1} \}$$

$$\text{lhs} = R_1[1, 20, 30] + R_1[1, 10, 31] + R_1[1, 11, 21] // .$$

$$\text{gRules}_{1,20,30} \cup \text{gRules}_{1,10,31} \cup \text{gRules}_{1,11,21};$$

$$\text{rhs} = R_1[1, 10, 20] + R_1[1, 11, 30] + R_1[1, 21, 31] // .$$

$$\text{gRules}_{1,10,20} \cup \text{gRules}_{1,11,30} \cup \text{gRules}_{1,21,31};$$

**Simplify[lhs == rhs]**

True

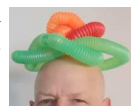
Next comes Reid1, where we use results from an earlier example:

$$R_1[1, 2, 1] - 1 (g_{22} - 1/2) / . \text{gRules}_{\alpha,\beta} \mapsto \begin{pmatrix} 1 & T^{-1} & 1 \\ 0 & T^{-1} & 1 \\ 0 & 0 & 1 \end{pmatrix} \llbracket \alpha, \beta \rrbracket$$

$$\frac{1}{T^2} - \frac{1}{T} - \frac{-1 + \frac{1}{T}}{T} = \text{loop}$$

Invariance under the other moves is proven similarly.

**Wearing my Topology hat** the formula for  $R_1$ , and even the idea to look for  $R_1$ , remain a complete mystery to me.



**Wearing my Quantum Algebra hat,** I spy a Heisenberg algebra  $\mathbb{H} = A\langle p, x \rangle / ([p, x] = 1)$ :

$$\text{cars} \leftrightarrow p \quad \text{traffic counters} \leftrightarrow x$$

**Where did it come from?** Consider  $\mathfrak{g}_\epsilon := \mathfrak{sl}_{2+}^\epsilon := L\langle y, b, a, x \rangle$  with relations

$$[b, x] = \epsilon x, \quad [b, y] = -\epsilon y, \quad [b, a] = 0,$$

$$[a, x] = x, \quad [a, y] = -y, \quad [x, y] = b + \epsilon a.$$

At invertible  $\epsilon$ , it is isomorphic to  $\mathfrak{sl}_2$  plus a central factor, and it can be quantized à la Drinfel'd [Dr] much like  $\mathfrak{sl}_2$  to get an algebra  $QU = A\langle y, b, a, x \rangle$  subject to (with  $q = e^{\hbar\epsilon}$ ):

$$[b, a] = 0, \quad [b, x] = \epsilon x, \quad [b, y] = -\epsilon y,$$

$$[a, x] = x, \quad [a, y] = -y, \quad xy - qyx = \frac{1 - e^{-\hbar(b+\epsilon a)}}{\hbar}.$$

Now  $QU$  has an  $R$ -matrix solving Yang-Baxter (meaning Reid3),

$$R = \sum_{m,n \geq 0} \frac{y^m b^m \otimes (\hbar a)^m (\hbar x)^n}{m! [n]_q!}, \quad ([n]_q! \text{ is a "quantum factorial"})$$

and so it has an associated "universal quantum invariant" à la Lawrence and Ohtsuki [La, Oh],  $Z_\epsilon(K) \in QU$ .

Now  $QU \cong \mathcal{U}(\mathfrak{g}_\epsilon)$  (only as algebras!) and  $\mathcal{U}(\mathfrak{g}_\epsilon)$  represents into  $\mathbb{H}$  via

$$y \rightarrow -tp - \epsilon \cdot xp^2, \quad b \rightarrow t + \epsilon \cdot xp, \quad a \rightarrow xp, \quad x \rightarrow x,$$

(abstractly,  $\mathfrak{g}_\epsilon$  acts on its Verma module

$$\mathcal{U}(\mathfrak{g}_\epsilon) / (\mathcal{U}(\mathfrak{g}_\epsilon)\langle y, a, b - \epsilon a - t \rangle) \cong \mathbb{Q}[x]$$

by differential operators, namely via  $\mathbb{H}$ ), so  $R$  can be pushed to  $\mathcal{R} \in \mathbb{H} \otimes \mathbb{H}$ .

Everything still makes sense at  $\epsilon = 0$  and can be expanded near  $\epsilon = 0$  resulting with  $\mathcal{R} = \mathcal{R}_0(1 + \epsilon \mathcal{R}_1 + \dots)$ , with  $\mathcal{R}_0 = e^{t(xp \otimes 1 - x \otimes p)}$  and  $\mathcal{R}_1$  a quartic polynomial in  $p$  and  $x$ . So  $p$ 's and  $x$ 's get created along  $K$  and need to be pushed around to a standard location ("normal ordering"). This is done using

$$(p \otimes 1)\mathcal{R}_0 = \mathcal{R}_0(T(p \otimes 1) + (1 - T)(1 \otimes p)),$$

$$(1 \otimes p)\mathcal{R}_0 = \mathcal{R}_0(1 \otimes p),$$

and when the dust settles, we get our formulas for  $\rho_1$ . But  $QU$  is a quasi-triangular Hopf algebra, and hence  $\rho_1$  is **homomorphic**. Read more at [BV1, BV2] and hear more at  $\omega\epsilon\beta/\text{SolvApp}$ ,  $\omega\epsilon\beta/\text{Dogma}$ ,  $\omega\epsilon\beta/\text{DoPeGDO}$ ,  $\omega\epsilon\beta/\text{FDA}$ ,  $\omega\epsilon\beta/\text{AQDW}$ .

Also, we can (and know how to) look at higher powers of  $\epsilon$  and we can (and more or less know how to) replace  $\mathfrak{sl}_2$  by arbitrary semi-simple Lie algebra (e.g., [Sch]). So  $\rho_1$  is **not alone!**



Schaveling

These constructions are very similar to Rozansky-Overbay [Ro1, Ro2, Ro3, Ov] and hence to the "loop expansion" of the Kontsevich integral and the coloured Jones polynomial.

If this all reads like **insanity** to you, it should (and you haven't seen half of it). Simple things should have simple explanations. Hence,

**Homework.** Explain  $\rho_1$  with no reference to quantum voodoo and find it a topology home (large enough to house generalizations!). Make explicit the homomorphic properties of  $\rho_1$ . Use them to do topology!