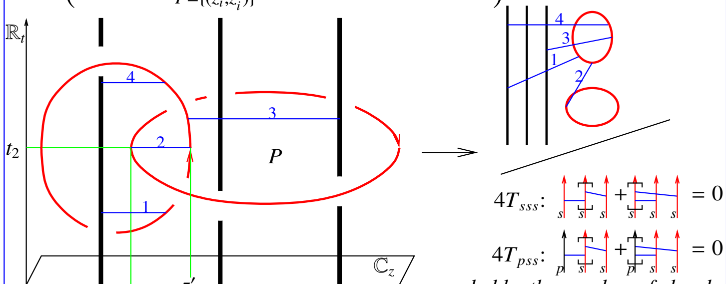
Kontsevich in a Pole Dance Studio. (w/o poles? See [Ko, BN])

$$Z=\left(\sum_{m=0}^{\infty}\frac{1}{(2\pi i)^m}\sum_{\substack{t_1<\ldots<t_m\\ P=\{(z_i,z_i')\}}}(-1)^{\#P_\downarrow}D_P\bigwedge_{i=1}^{m}\frac{dz_i-dz_i'}{z_i-z_i'}\right)^{\sim}\in\mathcal{A}$$



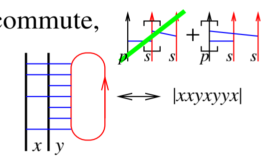
graded by the number of chords
filtered by the number of *ss* chords

Comments on the Kontsevich Integral.

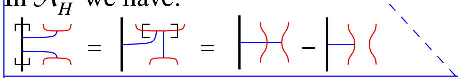
1. In the tangle case, the endpoints are fixed at top and bottom.
2. The $(\cdots)^{\sim}$ means "a correction is needed near the caps and the cups" (for the framed version, see [LM2, Da]).
3. There are never *pp* chords, and no $4T_{pps}$ and $4T_{ppp}$ relations.
4. Z is an "expansion".
5. Z respects the *ss* filtration and so descends to $Z^{/s}\colon \mathcal{K}^{/s}\to\mathcal{A}^{/s}$.

Comments on $\mathcal{A}$.

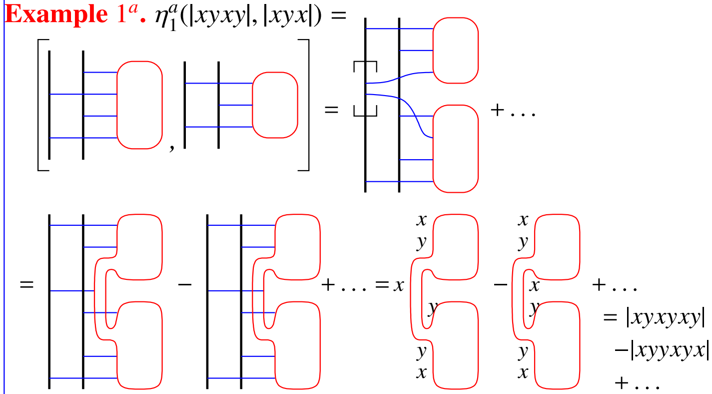
In $\mathcal{A}^{/1}$ legs on poles commute, so $\mathcal{A}^{/1}(\bigcirc)=|A|$!



In $\mathcal{A}_H^{/2}$ we have:

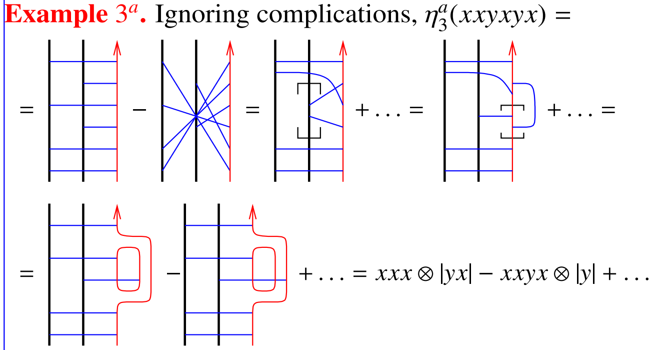


Example 1^a. $\eta_1^a(|xyxy|,|xyx|)=$



$= |xyxyxy| - |xyyxyx| + \ldots$

Example 3^a. Ignoring complications, $\eta_3^a(xxyxyx)=$



$+\ldots = xxx\otimes|yx| - xyx\otimes|y| + \ldots$

Proof of Lemma 1.

We partially prove Theorem 2 instead:

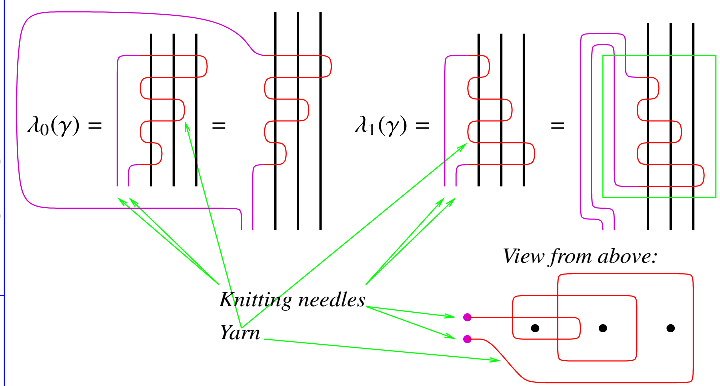
Theorem 2. $\mathrm{gr}^{\bullet}\mathcal{K}_H\cong\mathbb{F}[\![\hbar]\!]\otimes(\mathcal{K}^{/1})_0$.

Proof mod $\hbar^2$. The map $\leftarrow$ is obvious. To go $\rightarrow$, map $\mathcal{K}_H\to\mathbb{F}[\![\hbar]\!]\otimes\mathcal{K}^{/1}$ using ⤯ $\mapsto$ ✕ $+\frac{\hbar}{2}$)(and ⤰ $\mapsto$ ✕ $-\frac{\hbar}{2}$)(and apply the functor $\mathrm{gr}^{\bullet}$.

Unignoring the Complications.

We need λ_0 and λ_1 such that:

1. $\lambda_1(\gamma)$ is obtained from $\lambda_0(\gamma)$ by flipping all self-intersections from ascending to descending.
2. Up to conjugation, $\lambda_1(\gamma)$ is obtained from $\lambda_0(\gamma)$ by a global flip.
3. $Z(\lambda_i(\gamma))$ is computable from $W(\gamma)$ and $Z^{/1}(\lambda_i(\gamma))=W(\gamma)$.



Homework

1. Is there more than Examples 1–4?
2. Derive the bialgebra axioms from this perspective.
3. What more do we get if we don't mod out by HOMFLY-PT?
4. What more do we get if we allow more than one strand-strand interaction?
5. In this language, recover Kashiwara-Vergne [AKKN1, AKKN2].
6. How is all this related to w-knots?
7. Do the same with associators. Use that to derive formulas for solutions of Kashiwara-Vergne.
8. What's the relationship with the Habiro-Massuyeau invariants of links in handlebodies [HM] (same links, different filtration).
9. Pole dance on other surfaces!

Kashiwara Vergne

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References

[AKKN1] A. Alekseev, N. Kawazumi, Y. Kuno, and F. Naef, *The Goldman-Turaev Lie Bialgebra in Genus Zero and the Kashiwara-Vergne Problem,* Adv. Math. **326** (2018) 1–53, arXiv:1703.05813.

[AKKN2] A. Alekseev, N. Kawazumi, Y. Kuno, and F. Naef, *Goldman-Turaev formality implies Kashiwara-Vergne,* Quant. Topol. **11-4** (2020) 657-–689, arXiv:1812.01159.

[AN1] A. Alekseev and F. Naef, *Goldman-Turaev Formality from the Knizhnik-Zamolodchikov Connection,* Comp. Rend. Math. **355-11** (2017) 1138–1147, arXiv:1708.03119.

[BN] D. Bar-Natan, *On the Vassiliev Knot Invariants,* Top. **34** (1995) 423–472.

[Da] Z. Dancso, *On the Kontsevich Integral for Knotted Trivalent Graphs,* Alg. Geom. Topol. **10** (2010) 1317–1365, arXiv:0811.4615.

[HM] K. Habiro and G. Massuyeau, *The Kontsevich Integral for Bottom Tangles in Handlebodies,* Quant. Topol. **12-4** (2021) 593–703, arXiv:1702.00830.

[Ko] M. Kontsevich, *Vassiliev's Knot Invariants,* Adv. in Sov. Math. **16(2)** (1993) 137–150.

[LM1] T. Q. T. Le and J. Murakami, *Kontsevich's Integral for the HOMFLY Polynomial and Relations Between Values of Multiple Zeta Functions,* Top. and its Appl. **62-2** (1995) 193–206.

[LM2] T. Q. T. Le and J. Murakami, *The Universal Vassiliev-Kontsevich Invariant for Framed Oriented Links,* Comp. Math. **102-1** (1996) 41–64, arXiv:hep-th/9401016.

[Ma] G. Massuyeau, *Formal Descriptions of Turaev's Loop Operations,* Quant. Topol. **9-1** (2018) 39-–117, arXiv:1511.03974.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/LesDiablerets-2208/