Stonehenge

Witten, Stonehenge, Lie and Vassiliev

Arithmetic and Geometry Around Quantization, Istnabul , June 2006 (Lecture I)

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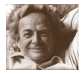
Witten

Feynman diagrams for the Chern–Simons–Witten theory:



Richard Feynman

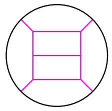
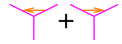
$$\int_{\mathfrak{g}\text{-connections}} \mathcal{D}A\, hol_K(A) \exp\left[\frac{ik}{4\pi}\int_{\mathbb{R}^3} \operatorname{tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)\right] \longrightarrow \sum_{\text{Feynman Diagrams } D} W_{\mathfrak{g}}(D) \sum\!\!\!\!\!\!\int \mathcal{E}(D) \longrightarrow \sum_{\text{Feynman Diagrams } D} D \sum\!\!\!\!\!\!\int \mathcal{E}(D)$$

When all the dust settles this becomes the generating function of all stellar coincidences:

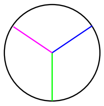
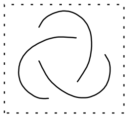
Dylan Thurston

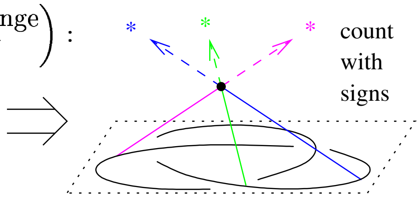
$$Z(K) := \lim_{N\to\infty} \sum_{3\text{-valent } D} \frac{1}{2^c c! \binom{N}{e}} \langle D, K\rangle_{\text{Stonehenge}} D \cdot \begin{pmatrix}\text{framing-}\\ \text{dependent}\\ \text{counter-term}\end{pmatrix} \in \mathcal{A}(\circlearrowleft)$$

N := # of stars
c := # of chopsticks
e := # of edges of D

$\mathcal{A}(\circlearrowleft)$:=Span ⟨  ⟩ / oriented vertices, AS:  =0 & more relations

$\langle D, K\rangle_{\text{Stonehenge}} :=$ (The signed Stonehenge pairing of D and K) :

$D=$  $K=$ 

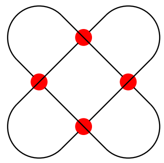


count with signs

The Gaussian linking number

$lk($  $) = \frac{1}{2}\sum_{\text{vertical chopsticks}}$ (signs)

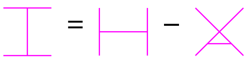
Carl Friedrich Gauss



$lk=2$

When deforming, catastrophes occur when:

A plane moves over an intersection point – Solution: Impose IHX,



(see below)

An intersection line cuts through the knot – Solution: Impose STU,

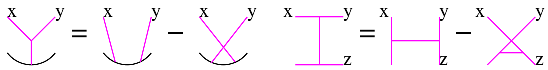
(similar argument)

The Gauss curve slides over a star – Solution: Multiply by a framing–dependent counter–term.

(not shown here)

Theorem. Modulo Relations, $Z(K)$ is a knot invariant!

Related to Lie algebras



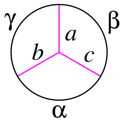
[x,y] = xy − yx   [[x,y],z]=[x,[y,z]]−[y,[x,z]]

Sophus Lie

More precisely, let $\mathfrak{g} = \langle X_a \rangle$ be a Lie algebra with an orthonormal basis, and let $R = \langle v_\alpha \rangle$ be a representation. Set

$$f_{abc} := \langle [a,b], c\rangle \qquad X_a v_\beta = \sum_\beta r^\beta_{a\gamma} v_\gamma$$

and then

$W_{\mathfrak{g},R}$:  $\longrightarrow \sum_{abc\alpha\beta\gamma} f_{abc} r^\beta_{a\gamma} r^\gamma_{b\alpha} r^\alpha_{c\beta}$

$W_{\mathfrak{g},R} \circ Z$ is often interesting:

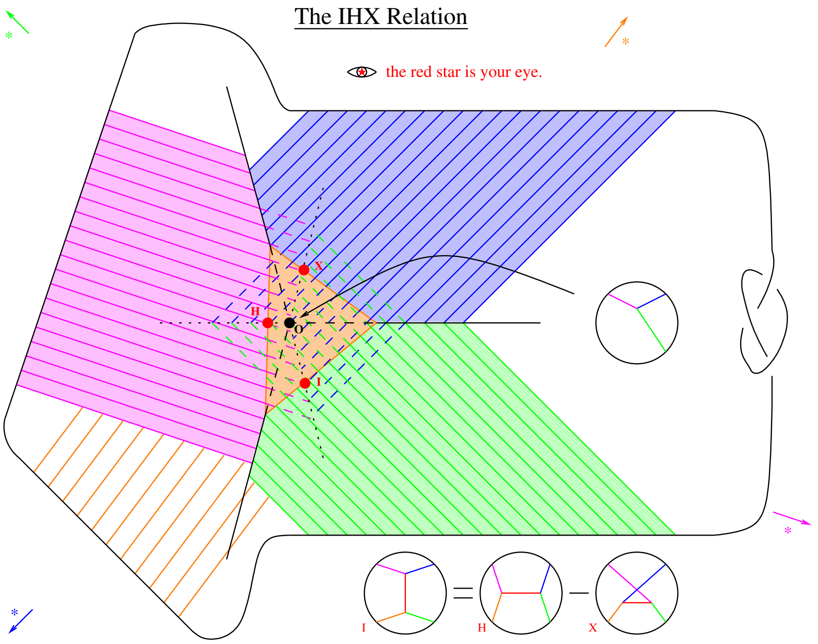
$\mathfrak{g} = sl(2)$ →  The Jones polynomial

$\mathfrak{g} = sl(N)$ →  The HOMFLYPT polynomial

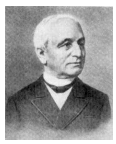
Przytycki

$\mathfrak{g} = so(N)$ →  The Kauffman polynomial

The IHX Relation



"God created the knots, all else in topology is the work of man."



Leopold Kronecker

(modified)

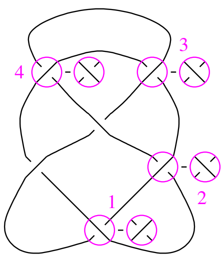
Definition. V is finite type (Vassiliev, Goussarov) if it vanishes on sufficiently large alternations as on the right

Theorem. All knot polynomials (Conway, Jones, etc.) are of finite type.

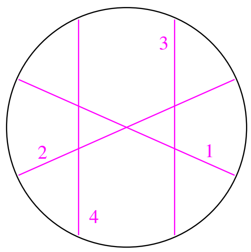
Conjecture. (Taylor's theorem) Finite type invariants separate knots.

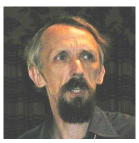
Theorem. $Z(K)$ is a universal finite type invariant!

(sketch: to dance in many parties, you need many feet).



The Miller Institute knot





Vassiliev

Goussarov

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