

Dror Bar-Natan: Talks: Hanoi-0708: **Following Lin: Expansions for Groups**



Riverside, April 2000



Kyoto, September 2001

- Vaughan's Hierarchy** (generalized, unauthorized)
- ☺ Computation
 - ☺ Formula
 - ☺ Proof
 - ☺ Theory
 - ☺ Dream

See Lin's "Power Series Expansions and Invariants of Links", 1993 Georgia International Topology Conference, AMS/IP Studies in Adv. Math. **2** (1997) 184-202.

The Magnan and Exponential Expansions

$$Z_{1,2} : G_n = \left(\begin{array}{c} \text{free group} \\ \text{on} \\ X_1, \dots, X_n \end{array} \right) \rightarrow \hat{A}_n = \left(\begin{array}{c} \text{completed free} \\ \text{associative} \\ \text{algebra on} \\ x_1, \dots, x_n \end{array} \right)$$

by $X_i \mapsto 1 + x_i$ or e^{x_i}

$$X_i^{-1} \mapsto 1 - x_i + x_i^2 - \dots \text{ or } e^{-x_i}.$$

What's "An Expansion"? A filtration-preserving isomorphism $Z : C(G) \rightarrow \mathcal{A}(G)$ where

$$I := \{ \sum a_i g_i : \sum a_i = 0 \} \subset CG$$

$$CG = I^0 \supset I^1 \supset I^2 \supset I^3 \supset \dots$$

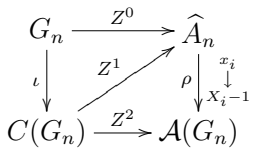
$$C(G) := \varprojlim_k CG/I^k \rightarrow \dots \rightarrow CG/I^2 \rightarrow CG/I \rightarrow 0$$

is filtered by $F_m C(G) := \varprojlim_{k>m} I^m/I^k$ and $\mathcal{A}(G) := \text{gr } C(G) = \bigoplus I^m/I^{m+1}$. So all expansions are equivalent!

Think duals! $C(G)^*$ are "finite type invariants". $\mathcal{A}(G)^*$ are "weight systems". Z is a "universal finite type invariant".

$Z_{1,2}$ are Expansions. With $Z^0 = Z_1$ or $Z^0 = Z_2$:

1. ι is automatic.
 2. ρ is well-defined.
 3. $Z^0|_{I^m} \subset F_m \mathcal{A}_n$.
 4. Z^0 descends to Z^1 .
 5. Define Z^2 .
 6. ρ is surjective.
 7. $\text{gr } Z^2$ is the identity.
 8. Z^2 is an isomorphism.
 9. ρ is an isomorphism.
- Everything generalizes, step 2 sometimes becomes tricky.



The Kontsevich Integral for Braids

Z_K

$$\sum_{\substack{m, t_1 < \dots < t_m \\ P = \{(z_i, z'_i)\}}} \frac{D_P}{(2\pi i)^m} \bigwedge_{i=1}^m \frac{dz_i - dz'_i}{z_i - z'_i}$$

$\mathcal{A} :=$

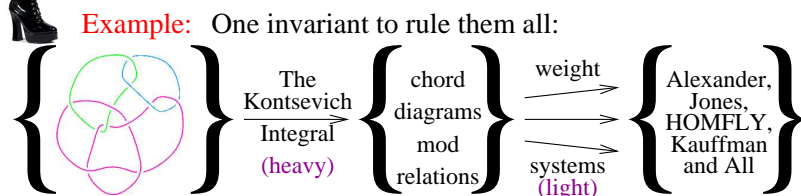
loc: $\begin{array}{|c|} \hline \text{---} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{---} \\ \hline \end{array}$

4T: $\begin{array}{|c|} \hline \text{---} \\ \hline \end{array} + \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} + \begin{array}{|c|} \hline \text{---} \\ \hline \end{array}$



Which other groups / groupoids / categories have expansions?

Dror's Dream / Obsession: "Unify" quantum groups – find one object that contains all.



Easy! Universal! A Morphism! Unique! An Isomorphism!
What is a "Quantum Group"? For now, a "deformation of the trivial" solution in $\mathcal{U}(\mathfrak{g})^{\otimes *}[[\hbar]]$ of the major equations:

$$(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta \quad R^{-1}\Delta R = \Delta^{op}$$

$$(\Delta \otimes 1)R = R^{23}R^{13} \quad (1 \otimes \Delta)R = R^{12}R^{13}$$

(as well as a few minor equations).

- Dror's Guess:** A unified object exists; we'll need:
1. Expansions as in Lin / universal finite type invariants.
 2. Naturality / functoriality.
 3. Knotted graphs, especially trivalent.
 4. Associators following Drinfel'd.
 5. The work of Etingof and Kazhdan on bialgebras.
 6. Virtual braids / knots / knotted graphs.
 7. Polyak (LMP 54) & Haviv (arXiv:math/0211031) on arrow diagrams. (and when construction ends, we'll dump the scaffolding)

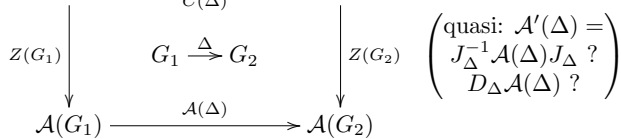
Why care? Quantum groups computable invariants make!

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(Quasi?) Natural Expansions

$G \mapsto C(G)$ and $G \mapsto \mathcal{A}(G)$ are functors. Can you choose a ((quasi?) natural) Z satisfying $C(G_1) \xrightarrow{C(\Delta)} C(G_2)$



Perhaps just on a subcategory of **Groups**? Perhaps **Braids** with strands addition, deletion and doubling:



Virtual Braids *crossings are real, strands go virtual*

Definition. Crossings, modulo Reidemeister moves, but the linkages between crossings are "virtual":

Polyak's $\vec{\mathcal{A}}$. modulo loc and "6T":

Lie bialgebras. The \mathfrak{g} in a sum $\mathfrak{g} \oplus \mathfrak{g}^*$ which in itself is a Lie algebra with subalgebras \mathfrak{g} and \mathfrak{g}^* , and in which the tautological metric is invariant. **Why bother?** Their deformations are quantum groups, and their diagrammatic universalization is $\vec{\mathcal{A}}$.

Question Can you interpret quantum groups as (quasi?)-natural expansions on virtual braids?

Dror's Guess: No, but the effort will be worthwhile.

