Abstract. To end a week of deep thinking with a nice colourful light dessert, we will present the Kolmogorov-Arnol’d solution of Hilbert’s 13th problem with lots of computer-generated rainbow-painted 3D pictures.

In short, Hilbert asked if a certain specific function of three variables can be written as a multiple (yet finite) composition of continuous functions of just two variables. Kolmogorov and Arnol’d showed him silly (ok, it took about 60 years, so it was a bit tricky) by showing that any continuous function \( f \) of any finite number of variables is a finite composition of continuous functions of a single variable and several instances of the binary function “+” (addition). For \( f(x, y) = xy \), this may be \( xy = \exp(\log x + \log y) \). For \( f(x, y, z) = x^y/z \), this may be \( \exp(\exp(\log y + \log \log x) + (- \log z)) \).

What might it be for (say) the real part of the Riemann zeta function?

The only original material in this talk will be the pictures; the math was known since around 1957.

Fix an irrational \( \lambda > 0 \), say \( \lambda = (\sqrt{5} - 1)/2 \). All functions are continuous.

Step 1. If \( \epsilon > 0 \) and \( f : [0, 1] \times [0, 1] \to \mathbb{R} \), then there exists \( \phi : [0, 1] \to [0, 1] \) and \( g : [0, 1 + \lambda] \to \mathbb{R} \) so that

\[
|f(x, y) - g(\phi(x) + \lambda\phi(y))| < \epsilon
\]

on at least 98% of the area of \([0, 1] \times [0, 1]\).

The key. “Poorify” chocolate bars.
Step 2. There exists $\phi : [0, 1] \to [0, 1]$ so that for every $\epsilon > 0$ and every $f : [0, 1] \times [0, 1] \to \mathbb{R}$ there exists a $g : [0, 1+\lambda] \to \mathbb{R}$ so that $|f(x, y) - g(\phi(x) + \lambda \phi(y))| < \epsilon$ on a set of area at least $1 - \epsilon$ in $[0, 1] \times [0, 1]$.

The key. “Iterated poorification”.

Step 3. There exist $\phi_i : [0, 1] \to [0, 1]$ (1 ≤ $i$ ≤ 5) so that for every $\epsilon > 0$ and every $f : [0, 1] \times [0, 1] \to \mathbb{R}$ there exists a $g : [0, 1+\lambda] \to \mathbb{R}$ so that

$$|f(x, y) - \sum_{i=1}^{5} g(\phi_i(x) + \lambda \phi_i(y))| < \left(\frac{2}{3} + \epsilon\right) \|f\|_{\infty}$$

for every $x, y \in [0, 1]$.

The key. “Shift the chocolates”...

...then iterate.

Exercise 1. Do the $m$-dimensional case.

Exercise 2. Do $\mathbb{R}^m$ instead of just $I^m$.

Bad Mouthing. I think low of PowerPoint!

- No good feelings towards Impress, Beamer, etc., either.
- Must sync with speaker, can’t look back at key points, don’t know what to look forward too.
- No upper bound to useless content.
- Nothing to take home.

Propaganda. I love handouts!

- I have nothing to hide and you can take what you want, forwards, backwards, here and at home.
- What doesn’t fit on one sheet can’t be done in one hour.
- It costs hours and pennies. The audience’s worth it!
- Can put hyperbolic geometry in every talk!