\[ \frac{dS_t}{S_t} = m dt + b dW_t \quad \text{risky asset} \]

\[ \frac{dB_t}{B_t} = r dt \quad \text{risk-free asset} \]

\[ g(t, S_t) \quad \text{price of claim that pays } \psi(S_T) \]

\[
\begin{align*}
\frac{d g}{dt} + rs \frac{d g}{d s} + \frac{1}{2} s^2 b^2 \frac{d^2 g}{d s^2} &= rg \\
\end{align*}
\]

\[ g(T, S) = \psi(S) \quad \text{Terminal condition} \]

BS PDE determines \( g \) uniquely.

![European call diagram](image)

To hedge (i.e. to have replicating portfolio),

\[ \xi_t = \frac{\partial g}{\partial S} (t, S_t) \quad \text{units of risky asset} \]
\[ q_t = \frac{\partial Q}{\partial S} (t, S_t) \] units of risky asset

\[ \beta_t = \frac{1}{\partial_s} \left( g(t, S_t) - S_t \frac{\partial g(t, S_t)}{\partial S} \right) \] units of risk-free asset

\[ \beta_t = e^{\alpha t} \]

In practice, we need to re-balance hedging portfolio finitely many times.

1) Time based hedging

\[ \Delta t = t_2 - t_1 = t_n - t_{n-1} \]

Find \( g \) by solving \( B-S \) PDE

To buy \( \Delta_0 = \frac{\partial g(0, S_0)}{\partial S} \) units of asset \( S \)

\[ M_0 = \beta_0 \Delta_0 = g(0, S_0) - S_0 \Delta_0 \] to bank account

\[ t_k \] buy \( \Delta_k = \frac{\partial g(t_k, S_{tk})}{\partial S} \) units of asset \( S \)
2) Move based hedging

\[ M_k = M_{k-1} e^{\text{rat}} + (\Delta k - \Delta u) S_k \]

bank account

collect interest

Same procedure for finding \( \Delta u, M_k \)

but choice of \( t_n \)'s

is different

"Delta hedging"

Recall: \( \frac{\partial g}{\partial S} = \text{delta of option} \)

 Hedging portfolio:

\[ V_t = d_t S_t + \beta_t \widehat{B}_t - g(t, S_t) \]

We have chosen \( d_t, \beta_t \) so that

\[ \Delta \pi = J \cdot \Delta S + \pi \Delta \rho \cdot -\alpha \Delta S \]
\[ V(t, S) = \Delta_t S + \beta_t e^{-r_t} - g(t, S) \]

satisfies

\[ V(t, S) = O((S - S_t)^2) \]

i.e., if \( S \approx S_t \) then \( V(t, S) \approx 0 \) \( \Rightarrow \Delta_t S + \beta_t e^{-r_t} \equiv g(t, S) \)

\text{Euro call}

\begin{align*}
\text{Note:} & \quad g(t, S) > \Delta_t S + \beta_t e^{-r_t} \\
& \quad g(t, S) > \Delta_t S + \beta_t e^{-r_t} \text{ if } S \neq S_t
\end{align*}

What if \( t \neq t_s \)

\[ -\Delta_t S_t - \beta_t B_t \]

\text{Value of this portfolio is always } \geq 0

\text{Arbitrage?}

\text{Explanation: } \frac{\partial g}{\partial t} < 0

\text{"theta"}
As time goes by
\[ g(t, S) \downarrow \] - Euro call

**Delta-gamma hedging**

\[ g(t, S) \quad - \quad \text{price of an option} \]

\[ \Gamma_t = \frac{\partial^2 g}{\partial S^2} \quad - \quad \text{"gamma" of the option} \]

**Goal:** to construct a better hedging portfolio

We'll need another option \( h(t, S) \)

\[ V_t = \begin{cases} 
2_t \text{ units } S_t \\
\beta_t \text{ units } A \text{ with free asset } B_t = e^{rt} \\
\gamma_t \text{ units } h \end{cases} \]

\[ V_t = 2_t S_t + \beta_t e^{rt} + \gamma_t h(t, S_t) \]

\[ g(t, S_t) \] - can find by solving
Apply Taylor formula at $S = S_t$:

$$V(t, S) = d_t S + \beta_t e^{r_t} + \gamma_t h(t, S) = d_t S + \beta_t e^{r_t} + \gamma_t \left( h(t, S_t) + \frac{dh}{ds}(S-S_t) \right) + \frac{1}{2} \frac{d^2 h}{ds^2} (S-S_t)^2$$

We have

$$g(t, S) = \frac{\partial g(t, S)}{\partial s}(S - S_t) + \frac{1}{2} \frac{d^2 g}{ds^2}(S - S_t)^2$$

Let's match $V(t, S)$ and $g(t, S)$ up to terms involving $(S - S_t)^2$ by choosing $d_t$, $\beta_t$, $\gamma_t$.
To derive the characteristics of the option, need k-1 other options.

Other greeks

"Vega": \( v_k = \frac{\partial g}{\partial \sigma} \) \( \sigma - \) volatility

European call: differentiable B-S formula to find

\[ \frac{\partial g}{\partial b} = SN'(d_1) \sqrt{T-t} > 0 \]

Since \( \frac{\partial g}{\partial b} > 0 \), the price of Euro call is monotonically increasing in \( b \) \( \Rightarrow \) there is 1-1 correspondence between option prices and vol.

"Rho": \( \frac{\partial g}{\partial r} \)

Barrier option: up-and-out

Option's value here \( \infty \)
Options on dividend paying assets

\[ \frac{dS_t}{S_t} = \mu dt + \sigma dW_t \]

\( \mu - \text{real world drift} \)

\( \sigma > 0 \quad \text{volatility} \)

Continuous payment of dividends:

at rate \( SS_t dt \)

As usual, assume

\[ \frac{dB_t}{B_t} = r dt \]

\( g(t, S_t) \) - price of option paying \( \varphi(S_T) \)

Redo B-S model:

\[ V_t = \beta_t S_t + \beta_t B_t - g_t + \int_0^t dV S u dudt \]

Require: \( V_0 = 0, \; dV_t = 0 \)
\[ dV_t = \zeta_t dS_t + \beta_t dB_t - dg_t + \zeta_t S_t dt \]

**Self-financing constraint**

\[ \zeta_t = \frac{\frac{\partial g}{\partial S} + \frac{1}{2} \frac{\partial^2 g}{\partial S^2} \cdot dS_t \cdot dS_t}{\frac{\partial g}{\partial S}} \]

\[ dV_t = \zeta_t (\mu S_t dt + b S_t dW_t) + \beta_t dB_t - \frac{\partial g}{\partial S} \left( \frac{\partial g}{\partial t} dt + \frac{\partial g}{\partial S} (\mu S_t dt + b S_t dW_t) + \frac{1}{2} \frac{\partial^2 g}{\partial S^2} \cdot b^2 S_t dt \right) + \zeta_t S_t dt \]

\[ = \zeta_t (\mu S_t dt + b S_t dW_t) + \beta_t dB_t - \left( \frac{\partial g}{\partial S} \left( \frac{\partial g}{\partial t} dt + \frac{\partial g}{\partial S} (\mu S_t dt + b S_t dW_t) + \frac{1}{2} \frac{\partial^2 g}{\partial S^2} \cdot b^2 S_t dt \right) + \zeta_t S_t dt \right) \]

\[ = \zeta_t (\mu S_t dt + b S_t dW_t) + \beta_t dB_t - \left( \frac{\partial g}{\partial S} \left( \frac{\partial g}{\partial t} dt + \frac{\partial g}{\partial S} (\mu S_t dt + b S_t dW_t) + \frac{1}{2} \frac{\partial^2 g}{\partial S^2} \cdot b^2 S_t dt \right) + \zeta_t S_t dt \right) \]

\[ = \left[ \zeta_t MS_t - \frac{\partial g}{\partial S} \frac{\partial g}{\partial S} MS_t - \frac{1}{2} \frac{\partial^2 g}{\partial S^2} \cdot b^2 S_t + \zeta_t dS_t \right] dt + \left[ \zeta_t b S_t - \frac{\partial g}{\partial S} b S_t \right] dW_t \]

\[ \Rightarrow \dot{S}_t = \frac{\partial g}{\partial S} \]
Ex.: Euro call: strike $K$, maturity $T$

PDE

\[ \frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV - rK e^{-rt} \]

Boundary condition:

\[ V(S, T) = \max(0, S - K) \]

Initial condition:

\[ V(S, 0) = \max(0, S - K) \]
$0 < t_1 < t_2 < \ldots < t_n < T$

Dividends are paid at $t_j$:

$a_j S_{t_j} = \quad 0 \leq a_j \leq 1
$

paid as dividend

$S_{t_j} = \lim_{t \to t_j} S_t$

Right after $t_j$, asset price will drop: (by no-arbitrage principle)

$S_{t_j} = (1 - a_j) S_{t_j}$

Ex: Euro call

$g(t, s) = S \bigcap_{t_j \leq t} \left[ (1 - a_j) N(d_1(t, t_j, s)) - \right.$

$\left. - r(t) N(d_2(t, t_j, s)) \right]$

Options on futures

1. Forward contract
\begin{align*}
\text{payoff: } \mathcal{X}(S_T) &= S_T - \kappa \\
\mathcal{F}_{t,\kappa}(T) &= \mathbb{E} \left[ e^{-r(T-t)} (S_T - \kappa) \right] \\
&= S_t - e^{-r(T-t)} \kappa
\end{align*}

2. Forward price $F_t(T)$ of asset $S_t$ at time $T$

\[
\text{strike } \kappa \text{ s.t. } f_{t,\kappa}(T) = 0
\]

\[
\Rightarrow F_t(T) = e^{r(T-t)} S_t
\]

HW: check that $F_t(T)$ is no arb. price of $S_t$ at time $t<T$

3. Futures contract: at each time step pays the movement in $F_t(T)$

\[
F_t(T) \quad 5 \quad 5.1 \quad 4.9 \\
\quad 0.1 \quad -0.2
\]

Costs nothing to enter futures contract,
only receive gains / pay losses

At $t = T$ receive in total: $S_T - F_0(T)$

For forward contracts:

'fair' strike $K = F_0(T)$ your payoff

$S_T - F_0(T)$

Difference between forward and futures contracts:

for futures payoff is spread over time.

4, Options on Futures.

$$\frac{dS_t}{S_t} = \mu dt + b dW_t$$

Forward price $F_t = S_t e^{r(T-t)}$
By Ito's lemma
\[
\frac{dF_t}{F_t} = (\mu - r)dt + b \, dw_t
\]

Risk-free asset

\[
\frac{dB_t}{B_t} = r \, dt
\]

Option: \( g(t, F_t) \) price of option paying \( \varphi(F_{T_0}) \)

\[ 
\text{To} \quad T \\
\uparrow \quad \Rightarrow \\
\text{futures expiry} \\
\text{option expiry} \\
\]

Let's find \( g \):

Portfolio: \( L_t \) units of \( F_t \) (futures contract)

\( \beta_t \) units of risk-free asset

\(-1\) option \( g_t \)

\[ V_t = \beta_t B_t - g_t \]

because futures contract is worth nothing
\[ dV_t = \lambda_t dF_t + \beta_t d\beta_t - dg_t = \]

\[ = \lambda_t \left( (m-r) F_t dt + b F_t dW_t \right) + \beta_t r \beta_t dt \]

\[ - \left( \frac{\partial g}{\partial t} dt + \frac{\partial g}{\partial F} dF + \frac{1}{2} \frac{\partial^2 g}{\partial F^2} dF : dF \right) \]

\[ = \lambda_t \left( (m-r) F_t dt + b F_t dW_t \right) + \beta_t r \beta_t dt \]

\[ - \left( \frac{\partial g}{\partial t} dt + \frac{\partial g}{\partial F} \left( (m-r) F_t dt + b F_t dW_t \right) \right) \]

\[ + \frac{1}{2} \frac{\partial^2 g}{\partial F^2} b^2 F_t^2 dt \]

\[ = \int \lambda_t (m-r) F_t + \beta_t r \beta_t - \frac{\partial g}{\partial t} - \frac{\partial g}{\partial F} (m-r) F_t + \]

\[ - \frac{1}{2} \frac{\partial^2 g}{\partial F^2} b^2 F_t^2 \] \[ dW_t + \right \left[ \lambda_t b F_t - \frac{\partial g}{\partial t} b F_t \right] dW_t \]

Want \[ dV_t = 0 \] \[ \Rightarrow \lambda_t = \frac{\partial g}{\partial F} \]

\[ \frac{\partial g}{\partial t} + \frac{1}{2} \frac{\partial^2 g}{\partial F^2} b^2 F_t^2 = \beta_t r \beta_t \]
\[ \text{Same } V_0 = 0, \text{ so } U_t = 0 \Rightarrow U_t = 0 \Rightarrow \beta_t' = \frac{g_t}{\beta_t} \]

Therefore,

\[
\begin{cases}
\frac{dg}{dt} + \frac{1}{2} b^2 F^2 \frac{d^2 g}{dr^2} = mg \\
g(T_0, F) = g(T_0)
\end{cases}
\]

---

**How to solve (10) numerically?**

\[
\begin{array}{c}
F_0 = 0 \quad F_1, F_2, F_3 \quad F = F_n \\
t_0 = 0 \quad t_1, t_2, t_3 \quad T = t_m
\end{array}
\]

\[ \Delta F = F_n - F_{n-1} \text{ same for all } n \]

\[ \Delta t = t_i - t_{i-1} \text{ same for all } i \quad F_n \]

\[ g(u) = g(t, F) \]

\[ \frac{dg}{dt} = g(\frac{g_{i,n} - g_{i-1,n}}{\Delta t}) \]
\[
\frac{\Delta g}{\Delta F} \approx \frac{g_{l,n+1} - g_{l,n}}{\Delta F}
\]

\[
\frac{\Delta^2 g}{\Delta F^2} \approx \frac{g_{l,n+1} - 2g_{l,n} + g_{l,n-1}}{(\Delta F)^2}
\]

\[
(t*) \quad \frac{g_{l,n} - g_{l-1,n}}{\Delta t} + \frac{1}{2} b^2 F_k \frac{g_{l,n+1} - 2g_{l,n} + g_{l,n-1}}{(\Delta F)^2} = \text{ Solve the linear system}
\]

\[
g_{l,n} \text{ known from terminal cond. } \phi(F_{D1})
\]

\[
g_{l-1,n} \text{ from (t*)}
\]