Dec 12 - Final exam
7-10 pm

Brownian motion

1. $W_t$ has continuous trajectories, $W_0 = 0$
2. $W_t - W_s \sim N(0, t-s)$
3. $W_t - W_s$ is independent of $W_u - W_v$, if $(s,t) \cap (v,u) = \emptyset$

Its integral
$$\int_0^t g(s, W_s) \, dw_s$$

Its lemma
$$f \in C^{1,2}$$

$$df = \frac{\partial f}{\partial t} \, dt + \frac{\partial f}{\partial W} \, dW_t + \frac{1}{2} \frac{\partial^2 f}{\partial W^2} \, dt$$

$$f(t, W_t) - f(0, W_0) = \int_0^t \frac{\partial f}{\partial s} \, ds + \int_0^t \frac{\partial f}{\partial W} \, dW_s + \int_0^t \frac{1}{2} \frac{\partial^2 f}{\partial W^2} \, ds$$
\[
\frac{d}{dt} f(x_t) = x_t^2
\]

By Ito's Lemma:

\[
f(W_t) - f(W_0) = \int_0^t W_s^2 \, dw_s + \frac{1}{2} \int_0^t 2W_s \, ds
\]

\[
\begin{align*}
W_t^3 &= W_0^3 = 0 \\
\int_0^t W_s^2 \, ds &= \frac{W_t^3}{3} - \int_0^t W_s \, ds
\end{align*}
\]

**SDE - Stochastic Diff. Eq.**

\[
\frac{dX_t}{dt} = b X_t \, dW_t
\]

\[
X_t = X_0 e^{-\frac{1}{2} b^2 t} + b W_t - GBM
\]
Ex.: \[ dS_t = \mu S_t dt + b S_t dw_t \]

\( \mu \in \mathbb{R} \) - drift

\( b > 0 \) - volatility

\( \mu = 0 \)
\[ dS_t = b S_t dw_t \]
\[ S_t = S_0 e^{(-\frac{1}{2}b^2)t + b W_t} \]

\( \mu > 0 \) contributes to growth of \( S_t \)

**Validity:** \( X_t = \ln S_t \)

Apply Itô’s Lemma: \( X_t = f(S_t) \)
\[ f(x) = \ln x \]

\[ dX_t = \frac{1}{S_t} dS_t + \frac{1}{2} \left( -\frac{1}{S_t^2} \right) d[S_t S_t]_t \]

Rewrite in integral form:
\[ ds_t ds_t = b^2 S_t^2 dt \]

\[ X_t - X_0 = \int_0^t \left( \mu S_u du + b S_u dw_u \right) - \frac{1}{2} \int_0^t b^2 du \]

\[ \ln S_t - \ln S_0 = t \left( \mu - \frac{1}{2} b^2 \right) - \frac{1}{2} \int_0^t b^2 du \]
\[ \ln S_t = \ln S_0 + \int_0^t \left( \mu - \frac{1}{2} \sigma^2 \right) du + \int_0^t \sigma dw_u \]

\[ (\mu - \frac{1}{2} \sigma^2) t + \sigma W_t \]

Take exp:

\[ S_t = S_0 e^{(\mu - \frac{1}{2} \sigma^2) t + \sigma W_t} \]

\[ E_x: (\text{Vasicek model}) \]

\[ (\star) \quad dr_t = \kappa (\theta - r_t) dt + \sigma dW_t \]

Continuous time version of

\[ r_n = r_{n-1} + \kappa (\theta - r_{n-1}) + \sigma Z \]

\[ Z \sim N(0,1) \]

Let's solve (\star).

\[ Y_t = e^{kt} \]

\[ r_t = f(t, r_t) \quad f(t, x) = e^{kt} \]

Apply Itô's lemma:

\[ \frac{df}{dt} = k e^{kt} \]

\[ \sigma^2 f = 0 \]

\[ \text{New Section 10 Page 4} \]
Apply Ito's lemma:

\[ dY_t = \kappa e^{-\kappa t} \int_0^t e^{-\kappa s} \, ds \, dt + \kappa e^{-\kappa t} \, dW_t \]

\[ dY_t = \kappa e^{-\kappa t} \int_0^t e^{-\kappa s} \, ds \, dt + \kappa (\theta - \mu) e^{-\kappa t} \int_0^t e^{-\kappa s} \, ds \, dt + \sigma dW_t \]

\[ dY_t = \kappa e^{-\kappa t} \int_0^t e^{-\kappa s} \, ds \, dt + \kappa \theta e^{-\kappa t} \int_0^t e^{-\kappa s} \, ds + \sigma dW_t \]

\[ Y_t - Y_0 = \int_0^t \kappa \theta e^{-\kappa s} \, ds + \int_0^t \kappa \theta e^{-\kappa s} \, ds + \sigma dW_t \]

\[ e^{\mu t} - e^{\mu r_0} = \int_0^t \kappa \theta e^{-\kappa s} \, ds + \int_0^t \kappa \theta e^{-\kappa s} \, ds + \sigma dW_t \]

Ornstein-Uhlenbeck (Mean-reverting) process

\[ r_t = r_0 + \kappa \theta \int_0^t e^{-\kappa (t-s)} \, ds + \theta \int_0^t e^{-\kappa (t-s)} \, ds + \sigma dW_t \]

- \( k \) - rate of mean reversion
- \( \theta \) - level of mean reversion
\[ dr_t = k(\theta - r_t)dt + \sigma dW_t \]

\[ \theta < r_t \Rightarrow dr_t < 0 \]
\[ \theta > r_t \Rightarrow dr_t > 0 \]

Why \( r_t \) is fundamentally different?

The entire trajectory of \( W_t \) is needed.

\[ r_t \neq h(t, W_t) \]
for any \( h(\cdot, \cdot) \)

\[ \begin{array}{c}
A_0 < C_0 < e^{rt} < C_1 < \ldots < C_n < C_d
\end{array} \]

Define \( C_0 = (1 + \beta_t)^{r_{t-1}} \) where

\[ \begin{cases} 
2 + A_n + \beta_t e^{rt} = C_n \\
2 + A_d + \beta_t e^{rt} = C_d
\end{cases} \]

Why not binomial trees?
Extend this pricing theory to continuous time setting.

\[
\begin{aligned}
\frac{dS_t}{S_t} &= \mu dt + \sigma dW_t \\
\text{\(\mu\)} &= \text{real world drift, \(\sigma > 0\) volatility}
\end{aligned}
\]

Our setup:

1. Option paying \(Y(S_T)\) at \(T > 0\) (maturity)

2. Risk free asset: \(dB_t = rB_t dt\) (ODE)

\[
B_t = e^{rt}, \quad B_0 = 1 - \text{assume}
\]

Goal: find \(g(t, S_t)\) \(t < T\), \(g \in C^{1,2}\)

\[
\text{price of the option at time } t \text{ for asset price } S_t
\]

Assumption: the price of the option is

A time form \(g(t, S_t)\) (i.e. only current time and asset price matter)
Note: "price" - no-arbitrage price. \( g(l, S_t) \) has to be such that there is no arbitrage in the economy.

Find \( g(l, \cdot) \).

Short sell the option -1 of \( g_t = g(t, S_t) \),
\[ d_t = S_t \]
\[ \beta_t \text{ of risk-free asset} \]

Portfolio:
\[ V_t = 2_t S_t + \beta_t e^{r_t} - g(t, S_t) \]

Let \( V_0 = 0 \). How does \( V_t \) change with time?

\[ dV_t = d(2_t S_t) + d(\beta_t e^{r_t}) - dg(t, S_t) \]

Makes sense from financial viewpoint, \( \beta_t \) but not mathematically.

\[ dV_t = 2_t dS_t + \beta_t dB_t - dg(t, S_t) \]
we should have extra terms: we set them to 0

"self-financing constant"

Now we apply Ito's Calculus:

\[ dV_t = dt \cdot ds_t + \beta_t \cdot dB_t - dg(t, S_t) = \]

\[ = 2t \left( mS_t \, dt + bS_t \, dw_t \right) + \beta_t \cdot rB_t \, dt \]

\[ - \left( \frac{\partial g}{\partial t} \frac{\partial g}{\partial S} \, ds_t + \frac{1}{2} \frac{\partial^2 g}{\partial S^2} \, ds_t \cdot ds_t \right) \]

\[ = \frac{b^2 S_t^2 \, dt}{2} \]

\[ = 2t \left( mS_t \, dt + bS_t \, dw_t \right) + \beta_t \cdot rB_t \, dt \]

\[ - \left( \frac{\partial g}{\partial t} \frac{\partial g}{\partial S} \left( mS_t \, dt + bS_t \, dw_t \right) + \frac{1}{2} b^2 S_t \frac{\partial^2 g}{\partial S^2} \, dt \right) \]

\[ = \left( 2t mS_t + \beta_t rB_t - \frac{\partial g}{\partial t} - mS_t \frac{\partial g}{\partial S} - \frac{1}{2} b^2 S_t \frac{\partial^2 g}{\partial S^2} \right) \, dt \]

\[ + \left( 2t bS_t - bS_t \frac{\partial g}{\partial S} \right) \, dw_t \]
\[ g - S_t \frac{\partial g}{\partial S} - \frac{\partial g}{\partial t} - \frac{1}{2} b^2 S_t^2 \frac{\partial^2 g}{\partial S^2} = B_t \]

\[ \begin{cases} 
\frac{\partial g}{\partial t} + r S \frac{\partial g}{\partial S} + \frac{1}{2} b^2 S^2 \frac{\partial^2 g}{\partial S^2} = r g \\
\psi(T, S) = \varphi(S) 
\end{cases} \]

Black-Scholes PDE

Note: real world drift \( \mu \) doesn't show up!

Hint: verify that if \( \varphi(S) = \max(S-K, 0) \)

then

\[ g(t, S) = SN(d_+(T-t, S)) - Ke^{-r(T-t)}N(d_-(T-t, S)) \]

\[ d_+(T-t, S) = \frac{1}{b(T-t)} \left( \ln \frac{S}{K} + \left( r + \frac{1}{2} b^2 \right) (T-t) \right) \]

For other \( \varphi \)'s we can solve B-S PDE numerically.

How do I hedge an option?
How to hedge an option?

European call: 
\[ \frac{dS}{dS} = N(d_+), \quad d_+ = \frac{\log(S_t/K) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \]

- hold $\frac{\partial g}{\partial S}$ units of asset $S$
- put $g(0, S_0) = S_t \frac{\partial g}{\partial S}$ to bank account

\[ B_t = K N(d_-), \quad d_- = \frac{\log(S_t/K) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \]

\[ \beta_+ \beta_- \]

\[ \frac{dg}{dS} \quad \text{if } t < T \]
\[ \ldots \quad \text{if } t \approx T \]

Limitations of B-S model

1. $r, \sigma > 0$ are constant - can be fixed
2. $S_t$ has only continuous trajectories
   - market crash = jump in $S_t$
   - can extend B-S model using e.g. Poisson processes

3. Transaction costs
3. Transaction costs

Continuous rebalancing of hedging portfolio (to have \( \frac{\partial g}{\partial t}(t, S_t) \) units of asset)

\( \implies \infty \) transaction costs

What to do:

\[
\begin{array}{c}
0 \text{ at } 20t \\
\uparrow \uparrow \uparrow \\
\text{rebalance only at these times} \\
\end{array}
\]

\( t = 0 \) sell option, get \( g(0, S_0) \)

buy \( L_0 \) units of risky asset,

\[
\left[ \frac{\partial g}{\partial S}(0, S_0) \right] \ 	ext{“Delta of the option”}
\]

costs \( L_0 S_0 \)

put remaining \( g(0, S_0) - L_0 S_0 \) to bank account.

\( t = 4t \) asset price have changed: \( S_t = S_{40t} \)
your position in the risky asset is worth $2_0 S_t$

get interest rate $M_o e^r$

Rebalance: $2_1 = \frac{\partial g}{\partial S}(S_{0} t, S_{0} t) \Delta t$ of risky asset

$M_1 = M_0 e^r + (2_0 - 2_1) S_1$

$t = k \Delta t$

$2_n$ with of risky asset, $2_n = \frac{\partial g}{\partial S}(t, S_n)$

$M_n = M_{n-1} e^r + (2_{n-1} - 2_n) S_n$

\[ g(t; S) \]

fix $t$

fix $t$

$L_t S + M_t$

value of our portfolio

according to $\beta$-

4. Infinite liquidity

In $B-S$ model: can buy or sell any number of
In B-S model: can buy or sell any number of shares without affecting the price.

\[
\frac{\partial g}{\partial S} > 0
\]

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Euro call

Alexander Schied