Two approaches to pricing theory:

1. $q(t, S_t) = \mathbb{E}^Q \left[ e^{-r(T-t)} \psi(S_T) \right]$

   - price at time $t$ for asset price $S_t$ at claim paying $\psi(S_T)$ at $t=T$

   (got this from CRR model)

Computing $\mathbb{E}^Q$ is equivalent to assuming $Z \sim N(0,1)$

$$
(r - \frac{1}{2} \sigma^2)(T-t) + \sigma(T-t) \cdot Z
$$

in

$$
S_T = S_t e^{(r - \frac{1}{2} \sigma^2)(T-t) + \sigma(T-t) \cdot Z}
$$

We can get this formula from SDE

$$
\text{d}S_t = r S_t \text{d}t + \sigma S_t \text{d}W_t
$$

$W_t$ - Brownian motion under $Q$

2. PDE approach

Solve
\[ \begin{align*}
\frac{\partial g}{\partial t} + rS \frac{\partial g}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 g}{\partial S^2} = rg \\
g(T, S) = \varphi(S)
\end{align*} \]

To get price \( g(t, S) \)

**Approach 1 (flexible):** we can assume

\[ dS_t = r(t) S_t dt + \sigma(t) S_t dW_t \]

or even

\[ dS_t = r(t) S_t dt + b(t, S_t) S_t dW_t \]

**Formula**

\[ g(t, S) = \mathbb{E} \left[ e^{-\int_t^T r(u) du} \varphi(S_T) \right] \]

still works (i.e., gives no arb. of option - next lecture)

**Feynman-Kac Formula**
\[ dS_t = a(t, S_t) dt + b(t, S_t) dW_t \]

Need to compute:

\[
\text{function } h(t, S) = \mathbb{E} \left[ H(S_T) \mid S_t = S \right]
\]

\(H\) - some function (view it as a 'profit')

Claim: \(h(t, S_t)\) is a martingale

What does 'martingale' mean?

\[
\mathbb{E}[h(t, S_t) \mid S_u] = h(u, S_u)
\]

\(u < t\)

\[ \mathbb{E} \left[ \int_0^t g(s) dw_s \right] = \int_0^t g(s) ds = 0\]

Martingale property of \(h\) to integral

Proof of claim:

\[
\mathbb{E}[h(t, S_t) \mid S_u] = \mathbb{E} \left[ \mathbb{E}[H(S_T) \mid S_t] \mid S_u \right]
\]
Apply this lemma to $h(t, S_t)$:

$$dh(t, S_t) = \delta_t h \ dt + \delta_S h \ dS + \frac{1}{2} \delta_{ss} h \ dS \cdot dS$$

$$= \begin{cases} 
\text{use SDE} & \Rightarrow \delta_t h \ dt + \delta_S h \left( a(t, S_t) dt + b(t, S_t) dW_t \right) \\
+ \frac{1}{2} \delta_{ss} h \left( b^2 dt \right) & = \left[ \delta_t h + a(t, S_t) \delta_S h + \\
+ \frac{1}{2} b^2(t, S_t) \delta_{ss} h \right] dt + b(t, S_t) \delta_S h \ dW_t \\
\text{drift} & \\
\text{Since } h(t, S_t) \text{ is a martingale } \Rightarrow \text{drift } = 0 \\
\text{Obtain that } \delta_t h + a(t, S_t) \delta_S h + \frac{1}{2} b^2(t, S_t) \delta_{ss} h = 0 \\
h \text{ must satisfy} \end{cases}$$
Feynman-Unc formula:

\[ dS_t = a(t, S_t) dt + b(t, S_t) dW_t \]

\[ h(t, S) = E \left[ H(S_T) \mid S_t = S \right] \]

satisfies

Feynman-Unc PDE

\[
\begin{align*}
\frac{\partial h}{\partial t} + a(t, S) \frac{\partial h}{\partial S} + \frac{1}{2} b^2(t, S) \frac{\partial^2 h}{\partial S^2} &= 0 \\
h(T, S) &= H(S)
\end{align*}
\]

Example: B-S PDE

\[ dS_t = \sigma S_t \frac{\partial h}{\partial S} \, dt + b \, S_t \, dW_t \]

\[ a(t, S) = r S \]

\[ b(t, S) = \sigma S \]

\[ g(t, S) = E \left[ e^{-r(T-t)} (S_T) \mid S_t = S \right] \]

From F-K:

\[ \frac{\partial g}{\partial t} + r S \frac{\partial g}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 g}{\partial S^2} = r g \]

Discounted Feynman-Unc:
\[ h(t; S) = \mathbb{E} \left[ e^{-r(T-t)} H(S_T) \mid S_t = s \right] \]

\[
\begin{cases}
\frac{\partial}{\partial t} h + a(t, S) \frac{\partial}{\partial S} h + \frac{1}{2} b^2(t, S) \frac{\partial^2}{\partial S^2} h = r h \\
h(T, S) = H(S)
\end{cases}
\]

**Multiple Brownian motions**

\[ \bar{W}_t = (W^1_t, \ldots, W^d_t) \]

where \( W^i_t \) is a Brownian motion, \( W^i_t \) is independent of \( W^j_t \) if \( i \neq j \).

**Def.:** Cross-variation of \( X_t, Y_t \)

\[ [X, Y]_t = \lim_{\text{diam}(T) \to 0} \sum (X_{t+u} - X_t)(Y_{t+u} - Y_t) \]

\[ [X, X]_t \text{ called quadratic variation} \]

\[ D_{ij} = \begin{cases} 
1 & \text{if } i = j \\
0 & \text{if } i \neq j
\end{cases} \]
Proposition: \[ [W_i, W_i]_t = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \]

Proof:
Show: \[ \mathbb{E} \left( \sum (W_{tu}^i - W_{tu}^j)(W_{tu+i}^i - W_{tu+i}^j) \right) \to 0 \]

Assume \( i \neq j \):
\[ \text{Var} \left[ \sum (W_{tu+i}^i - W_{tu}^i)(W_{tu+i}^j - W_{tu}^j) \right] \to 0 \]

(already known that \( [W_i, W_i]_t = 1 \))

Since \( W_{tu}^i \) and \( W_{tu}^j \) are independent, \( W_{tu+i}^i - W_{tu}^i \) and \( W_{tu+i}^j - W_{tu}^j \) are independent as well.

\[ \mathbb{E} \left( \sum (\cdots)(\cdots) \right) = \sum \mathbb{E} \left( W_{tu+i}^i - W_{tu}^i \right) \mathbb{E} \left( W_{tu+i}^j - W_{tu}^j \right) = 0 \]

\[ \text{Var} \left[ \sum (\cdots)(\cdots) \right] = \mathbb{E} \left[ \left( \sum (\cdots)(\cdots) \right)^2 \right] = t_{tu+i} \]
\[ |\text{calculation}| = \mathbb{E} \left[ \sum_i (\ldots)^2 (\ldots)^2 \right] = \sum_i \mathbb{E} \left( W_{t_{n+1}}^i - W_{t_n}^i \right)^2 \]

\[
= \sum_i (t_{n+1} - t_n)^2 \leq \text{diam}(r) \sum_i (t_{n+1} - t_n) = \text{diam}(r) \cdot T \rightarrow 0
\]

as \( \text{diam}(T) \rightarrow 0 \)

**Done!**

In differential notation:

\[
dW_t^i \cdot dW_t^i = \begin{cases} 0 & \text{if } i \neq j \\ dt & \text{if } i = j \end{cases}
\]

**Itô Lemma for 2D Brownian Motion**

\[
\bar{W}_t = (W_t^1, W_t^2)
\]

\[
X_t = X_0 + \int_0^t f_i(s, W_s^1, W_s^2) \, ds + \int_0^t b_{1i}(s, W_s^1, W_s^2) \, dW_s^1 + \int_0^t b_{2i}(s, W_s^1, W_s^2) \, dW_s^2
\]
\[ Y_t = Y_0 + \int_0^t f_2(s, w_s^1, w_s^2) ds + \]
\[ + \int_0^t b_{21}(s, w_s^1, w_s^2) dw_s^1 + \int_0^t b_{22}(s, w_s^1, w_s^2) dw_s^2 \]

In \( d t, d t \) notation:

\[ d X_t = f_1(t, w_t^1, w_t^2) dt + b_{11}(t, w_t^1, w_t^2) dw_t^1 \]
\[ + b_{12}(t, w_t^1, w_t^2) dw_t^2 \]

\[ d Y_t = f_2(t, w_t^1, w_t^2) dt + b_{21}(t, w_t^1, w_t^2) dw_t^1 \]
\[ + b_{22}(t, w_t^1, w_t^2) dw_t^2 \]

\( dw_t^1 \cdot dw_t^2 = 0 \) - known

\[ [X, X]_t = \int_0^t (b_{11}^2 + b_{12}^2) ds \leq \frac{d X_t \cdot d X_t}{t} = \frac{\|b_{11} + b_{12}\|^2}{t} \]

\[ [Y, Y]_t = \int_0^t (b_{21}^2 + b_{22}^2) ds \leq \frac{d Y_t \cdot d Y_t}{t} \leq \frac{(\|b_{21}\|^2 + \|b_{22}\|^2)}{t} \]

\[ [X, Y]_t = \int_0^t (b_{11} b_{21} + b_{12} b_{22}) ds \]
Itô Lemma for 2D BM: \( f \in C \)

\[ \begin{align*}
  df(t, x_t, y_t) &= \frac{\partial}{\partial t} f dt + \frac{\partial}{\partial x} f dx_t + \frac{\partial}{\partial y} f dy_t \\
  &+ \frac{1}{2} \frac{\partial^2}{\partial x^2} f dx_t dx_t + \frac{1}{2} \frac{\partial^2}{\partial x \partial y} f dx_t dy_t + \frac{1}{2} \frac{\partial^2}{\partial y^2} f dy_t dy_t 
\end{align*} \]

Feynman-Kac formula for 2D BM:

\[ \begin{align*}
  g(t, x, y) &= \mathbb{E} \left[ N(x_T, y_T) \mid x_t = x, y_t = y \right] \\
  h(t, x, y) &= \mathbb{E} \left[ e^{-r(T-t)} N(x_T, y_T) \mid x_T = x, y_T = y \right] 
\end{align*} \]

Then

\[ \begin{align*}
  d_t g &= f_1 \frac{\partial}{\partial x} g + f_2 \frac{\partial}{\partial y} g + \frac{1}{2} \left( b_{11}^2 + b_{22}^2 \right) \frac{\partial^2}{\partial x^2} g \\
  &+ \left( b_{11} b_{22} + b_{12} b_{21} \right) \frac{\partial}{\partial x \partial y} g \\
  &+ \frac{1}{2} \left( b_{11}^2 + b_{22}^2 \right) \frac{\partial^2}{\partial y^2} g \\
  g(t, x, y) &= N(x, y)
\end{align*} \]
\[ \ldots = rh \]

\[ h(T, X, Y) = H(X, Y) \]

**Correlated stocks**

\[
\begin{cases}
\frac{dS^1_t}{S^1_t} = \mu dt + b_1 dW^1_t \\
\frac{dS^2_t}{S^2_t} = \mu dt + b_2 (\rho dW^1_t + \sqrt{1-\rho^2} dW^2_t)
\end{cases}
\]

\[ W^1_t, W^2_t \text{ are indep. } -1 \leq \rho \leq 1 \]

\[ \tilde{W}^2_t = \rho W^1_t + \sqrt{1-\rho^2} W^2_t, \quad \tilde{W}^1_t = W^1_t \]

**Brownian motion**

**Equivalent notation**

\[ \int dS^1_t = \int d\tilde{W}^1_t \]

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\[
\begin{align*}
\frac{dS_t^1}{S_t^1} &= \mu_1 dt + b_1 d\tilde{W}_t^1, \\
\frac{dS_t^2}{S_t^2} &= \mu_2 dt + b_2 d\tilde{W}_t^2
\end{align*}
\]

\[
d\tilde{W}_t^1 d\tilde{W}_t^2 = dw_t^1 \cdot (p dw_t^1 + \sqrt{1-p^2} dw_t^2) =
\]

\[
= p \ dw_t^1 dw_t^1 = p \ dt
\]

\[
(\tilde{W}_t^1, \tilde{W}_t^2 \text{ are not independent})
\]

**Example:** B-S formula for correlated stocks

Assume that under risk-neutral measure \( \tilde{Q} \)

\[
\begin{align*}
\frac{dS_t^1}{S_t^1} &= r dt + b_1 d\hat{W}_t^1, \\
\frac{dS_t^2}{S_t^2} &= \rho dw_t^1 + \sqrt{1-\rho^2} dw_t^2
\end{align*}
\]

\( \hat{W}_t^1, \hat{W}_t^2 \text{ are BM under } \tilde{Q} \)
\( \hat{W}_t, \hat{W}_t \) one indep. BMs under \( \mathbb{Q} \)

Price of option \( g(t, S_t, S_t^2) = \mathbb{E}^{\mathbb{Q}} \left[ e^{-r(t+T)} \psi(S_T, S_T^2) \right] \)

Apply Feynman-Kac (discounted)

\[
\begin{align*}
\partial_t g + r S_t \partial_{S_t} g + r S_t^2 \partial_{S_t^2} g + \frac{1}{2} b_1(S_t^1)^2 \partial_{S_t^1 S_t^1} g + \\
+ \left( b_1 S_t^1 b_2 p S_t^2 + 0 \right) \partial_{S_t^1 S_t^2} g + \\
- \frac{1}{2} \left[ b_1^2 g(S_t^1)^2 + b_2 \left( 1 - p^2 \right) (S_t^2)^2 \right] \partial_{S_t^2 S_t^2} g = r g
\end{align*}
\]

(\textbf{B-S PDE for } \( S_t^1, S_t^2 \) )

\( g(t, S_t, S_t^2) = \psi(S_t, S_t^2) \)

\textbf{Asian options}

\( S_t \) - stock price
TWAP (time-weighted average price)

\[ A_t = \frac{1}{t} \int_0^t S_u \, du \]

Can have options on \( A_t \), e.g. \( V(A_T) = \max (A_T - K, 0) \)

(reason: to prevent stock price manipulation affecting option prices close to maturity)

Our framework:

1) \[ \frac{dS_t}{S_t} = \mu dt + \sigma dW_t \quad \text{GBM} \]

2) \[ dA_t = \left( -\frac{dt}{t^2} \right) \int_0^t S_u \, du + \frac{1}{t} S_t \, dt \]

\[ dA_t = \frac{1}{t} S_t \, dt - \frac{1}{t^2} A_t \, dt \]

Rewrite:

\[ \begin{aligned}
    dS_t &= \mu S_t \, dt + \sigma S_t \, dW_t \\
    dA_t &= \frac{1}{t} S_t \, dt - \frac{1}{t^2} A_t \, dt
\end{aligned} \]
Have Feynman-Vac formula for this. More generally, for systems of SDEs:

\[
\begin{aligned}
    dS_t^1 &= \mu_1 S_t^1 + \mu_2 S_t^2 + b_1 dW_t^1 \\
    dS_t^2 &= \mu_2 S_t^1 + \mu_2 S_t^2 + b_2 \left( p dW_t^1 + \sqrt{1 - p^2} dW_t^2 \right)
\end{aligned}
\]

HW: Use Itô's lemma to derive Feynman-Vac form, convert pricing PDE for Asian options.

**Risky neutral measure**

Multidimensional market model:

\[
    dS_t^i = \mu_i S_t^i dt + \sum_{k=1}^i \sigma_{ik} S_t^i dW_t^k
\]

\[i = 1, \ldots, m\]

\[W_t^1, \ldots, W_t^m\] and pairwise indep.
\( P \) - real world prob. measure

**Def.:** \( P, Q \) prob. measures are equivalent

\[ P(A) = 0 \iff Q(A) = 0 \]

**Def.:** \( Q \) is risk neutral measure if it is equiv. to \( P \)

| \( E \) | \( e^{r T} S_t | S_u \) | \( e^{r T} S_u \) |
|---|---|---|
| (martingale condition) | \( i = 1, \ldots, m \) |

**Def.:** An arbitrage is a portfolio \( V_t \) s.t.

\[ V_0 = 0 \] and some \( t \)

\[ P[V_t > 0] = 1, \quad P[V_t > 0] > 0 \]

\[ \exists \text{ arbitrage} \iff \exists \text{ } U_t \text{ s.t. } U_0 > 0 \text{ and s.t. } \]

\[ P[U_t > 0] = 1, \quad P[U_t > 0] > 0 \]
\[ P \left[ U_t > e^{rt} U_0 \right] = 1 \quad P \left[ U_t > e^{rt} U_0 \right] = 0 \]

First Fundamental Theorem of Asset Pricing:

A market model does not admit arbitrage iff \( \mathcal{F} \) risk neutral measure \( \mathbb{Q} \)

When does \( \mathbb{Q} \) exist?

Rewrite eqs for \( S_t^i \)

\[ d \left( e^{-rt} S_t^i \right) = \exp \left( dS_t^i - rS_t^i \, dt \right) = \]

Ito lemma

\[ = e^{-rt} \left( \int m_i S_t^i \, dt + \sum_{i=1}^\infty \frac{d}{d} \left( b_i u \, dW_t^u - rS_t^i \, dt \right) = \right. \]

\[ = e^{-rt} S_t^i \left( \left( m_i - r \right) dt + \sum_{i=1}^\infty b_i u \, dW_t^u \right) \]

if we can rewrite

\[ = e^{-rt} S_t^i \left( \sum_{i=1}^\infty b_i u \, \Theta_k dt + \sum_{i=1}^\infty b_i u \, dW_t^u \right) \]
\[ e^{r_t} S_t \sum_{i=1}^{d} \left( \theta_i k_t + d \tilde{W}_t^u \right) \]

can change measure P to Q

to make \( \tilde{W}_t^u \)

a martingale,

and Q is a risk neutral measure

\[
\sum_{i=1}^{d} b_i \theta_i = \mu_i - r
\]

(m equations, d unknowns)

"market price of risk equations"

\( m \) is small, \( d \) is large \( \Rightarrow \) might get too many \( \theta_i \)’s

\( \Rightarrow \) infinitely many Q’s

\( m \) is large, \( d \) is small

\( \Rightarrow \) could be but no \( \theta_i \)’s exist, i.e., no Q.