Review

1) Binomial trees (CRR model)

2) Black-Scholes formula

\[
\begin{align*}
\text{Black-Scholes PDE} & \quad \left\{ \begin{array}{l}
\text{GBM asset price} \\
\text{European call, put}
\end{array} \right.
\end{align*}
\]

3) Any PDE, any payoff

Feynman-Kac PDE - use to compute

\[
\mathbb{E} \left[ e^{-rT} \mathbb{Q}(S_T) \right]
\]

(Price of option paying \(\mathbb{Q}(S_T)\))

risk neutral probability measure
Risk-neutral probability measure

One stock, one BM

\[ \frac{dS_t}{S_t} = \mu(t) dt + b(t) dw_t \]

deterministic functions

\[ \frac{dB_t}{B_t} = -R(t) dt \quad \Rightarrow \quad B_t = e^{-\int_0^t R(s) ds} \]

Minimum rate if \( R(t) = r \) then

\[ B_t = e^{-rt} \]

Option paying \( \varphi(S_T) \) at \( t = T \)

Claim: The no-arbitrage price (replicating price, hedging price)

\[ E_0^Q \left[ e^{-\int_0^T R(s) ds} \varphi(S_T) \right] = E_0^Q \left[ B(T) \varphi(S_T) \right] \]

at \( t = 0 \)
\[ B(t) g(t, S) = \mathbb{E} \left[ B(t) \varphi(S_t) \mid S_t = S \right] \]

Can use Feynman-Kac to find \( g \). Done!

Why this claim is true?

Recall: \( Q \) is risk neutral if

1) \( Q \sim P \) (i.e., \( B(A) = 0 \iff Q(A) = 0 \))

2) \( B(t) S_t \) is a martingale under \( Q \)

\[ \mathbb{E}[^{Q} \left[ B(t) S_t \mid S_n \right] = B(u) S_n \]

\[ u < t \]

\[ 0 < q < 1 \] risk neutral if \( A_g = \frac{1}{1+r} \left[ q A_t + (1-q) A_d \right] \]

i.e. \( A \) is a martingale

Asset price \( S_t \) under \( Q \):

\[ dS_t = \mu(t) S_t dt + \sigma(t) S_t dW_t \] under \( P \)
\[ d(B(t)S_t) = -R(t)B(t)S_t \, dt + B(t) \, dS_t = \]
\[ = -R(t)B(t)S_t \, dt + B(t) \left( \mu(t)S_t \, dt + \sigma(t)S_t \, dW_t \right) \]
\[ = (\mu(t) - R(t))B(t)S_t \, dt + \sigma(t)B(t)S_t \, dW_t \]
\[ = \theta(t)B(t)S_t \left( dW_t + \theta(t) \, dt \right) \]

\[ \theta(t) = \frac{\mu(t) - R(t)}{\sigma(t)} \quad \text{market price of risk} \]

\[ \text{excessive return per unit of risk} \]

Can change \( \mathbb{P} \) to another prob. measure \( \mathbb{Q} \) to make

\[ dW^\mathbb{Q}_t = dW_t + \theta(t) \, dt \]

where \( W^\mathbb{Q}_t \) is a BM under \( \mathbb{Q} \)

Girsanov theorem: such \( \mathbb{Q} \) equivalent to \( \mathbb{P} \) exists
\[ \Rightarrow \mathbb{E}^\mathbb{Q} [B(t)S_t] = B(0)S_0 \]

more generally
\[ \mathbb{E}^\mathbb{Q} [B(t)S_t] = B(0)S_0 \]

i.e. discounted asset price is a martingale under \( \mathbb{Q} \).

We didn't have any options yet

why [ ] formula is true

Consider a portfolio \( V_t \) which consists of \( dt \) units of asset \( S_t \), the rest is put in a bank account

\[ B(t)V_t \text{ under risk-neutral prob. measure } \mathbb{Q}. \]

\[ dV_t = \underbrace{\frac{d}{dt} B(t)S_t}_{\text{self-financing}} + R(t) B(t) \left( V_t - dS_t \right) dt \]
\[ d ( B(t) V_t ) = \Delta_t B(t) B(t) S_t \, d \tilde{W}_t \]

Integrate LHS, RHS, get Itô integral ⇒

\[ \mathbb{E} [ B(t) V_t ] = B(0) V_0 \quad (B(0) V_n) \]

⇒ \( B(t) V_t \) is a martingale under \( \mathbb{Q} \)

for any 'trading strategy' \( \Delta_t \)

By Martingale Representation theorem

\[ B(t) V_t = B(0) V_0 + \int_0^t \Gamma_u \, d \tilde{W}_u \]

where \( \Gamma_t \) is some process

Compare with

\[ B(t) g(t, S_t) = \mathbb{E}^\mathbb{Q} \left[ B(T) \varphi(S_T) \mid S_t \right] \]

\[ \quad \quad \text{definition} \]

Can we choose \( V_t = g(t, S_t) \)?
\[ B(t) V_t = \mathbb{E}^Q [ B(T) \Psi(S_T) | S_t ] \]

By:

\[ B(t) V_t = B(0) V_0 + \int_0^t d_u b(u) B(u) S_u d\tilde{W}_u \]

Combine:

\[ p_u = 2 u b(u) B(u) S_u \]

Choose in this way, i.e. \( \Phi_t = \frac{p_t}{\mathbb{E}_t[b(t)B(t)]} \)

\[ \Rightarrow \text{ get } V_t = g(t, S_t) \]

i.e. \( g(t, S_t) \) \( : \) hedging (or replicating) price of the option.

Presentation of option price as an expectation is useful because now you can run Monte Carlo simulations.

Prop: risk neutral prob measure \( \mathbb{Q} \Rightarrow \) no arbitrage

Proof: Suppose \( \exists V_t \) - arbitrage portfolio
\[ V_0 = 0 \quad \text{and at some } t \quad V_t \geq 0 \quad \text{with probability } P(V_t > 0) > 0 \]

Under \( Q \), \( V_t \) is a martingale

\[ E^Q [B(t)V_t] = B(0)V_0 = 0 \]

On the other hand,

\[ E^Q [B(t)V_t] > 0 \quad \text{Contradiction} \]

In multidimensional market model (previous lecture), it could be that there is no probability measure that is risk neutral.

\[ \Rightarrow \exists \text{ arbitrage} \quad \text{don't use this model!} \]

or there might be many risk neutral prob. measures.

\[ \Rightarrow \exists \text{ options that can not be hedged (i.e. no replicating portfolio for each option)} \]

Def: If there exists only one risk neutral prob. measure, then the market is called complete.
measure, then the market is called complete.

Asian options (cont'd)

Option paying \( \frac{1}{T} \int_0^T S_u \, du \) at \( t = T \)

\[ S_t - \text{asset price} \]

\[
\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \quad \text{under } \mathbb{P}
\]

\[
\frac{dS_t}{S_t} = rd t + \sigma \widetilde{dW}_t \quad \text{under } \mathbb{Q}
\]

\[
\begin{aligned}
\text{equivalent to } &
\end{aligned}
\]

\[
\begin{aligned}
\mathbb{E} [ e^{-rT} S_T ] &= e^{-ru} S_u \\
&\text{because } \quad (r - \frac{1}{2} \sigma^2) t + \sigma \widetilde{W}_t \\
S_t &= S_0 e^{rt}
\end{aligned}
\]
New variable: \( \text{variant 1} \)

\[ A_t = \frac{1}{t} \int_0^t S_u \, du \]

\( \text{variant 2} \)

\[ A_t = \int_0^t S_u \, du \]

Payoff \( \varphi \left( \frac{1}{t} A_t \right) \)

Need an SDE for \( A_t \).

\[ \begin{cases} 
  dA_t = S_t \, dt \\
  dS_t = r S_t \, dt + b \, dW_t 
\end{cases} \]

The price \( g(t, S, A) \) of the option paying \( \varphi \left( \frac{1}{t} A_t \right) \) at \( t = T \) will satisfy

\[ e^{-rT} g(t, S, A) = \mathbb{E}_Q^F \left[ e^{-rT} \varphi \left( \frac{1}{T} A_T \right) \mid S_T = S, A_T = A \right] \]

Merton's model for \( S_T, A_T \):

\[ g(t, S, A) = \mathbb{E}_Q \left[ e^{-r(T-t)} \varphi \left( \frac{1}{T-t} A_T \right) \mid S_t = S, A_t = A \right] \]
\[ d\left( e^{-rt} g(t, S_t, A_t) \right) = (\ldots) dt + (\ldots) dW_t \]

using Itô's lemma 2D

\[ \text{drift = 0} \] because

\[ e^{-rt} g(t, S_t, A_t) \]

is a martingale

(Alternatively, use Feynman-Kac)

Use "drift = 0":

\[ d\left( e^{-rt} g(t, S_t, A_t) \right) = -re^{-rt} g dt + e^{-rt} dg = \]

\[ = -re^{-rt} g dt + e^{-rt} \left( \delta_t g + \delta_s g dS_t + \delta_A g dA_t + \tfrac{1}{2} \delta_{ss} g dS_t^2 dS_t + \delta_{AA} g dA_t^2 + \tfrac{1}{2} \delta_{AA} g dA_t dA_t \right) \]

\[ ds dA = \left( rs_t dt + b S_t dW_t \right) \cdot S_t dt = 0 \]

\[ dt dA = \left. S_t S_t dt dt \right|_{dt = 0} = 0 \]

\[ dA dA = \left. S_t S_t dt dA \right|_{dt = 0} = 0 \]

\[ \Rightarrow -re^{-rt} g dt + e^{-rt} \left( \delta_t g + \delta_s g r S_t dt + \delta_A g S_t dt \right. \]

\[ + \tfrac{1}{2} \delta_{ss} g b^2 S_t^2 dt \bigg) + e^{-rt} \delta_{ss} g b S_t dW_t \]
\[
+ \frac{1}{2} \sigma^2 \dd s g \cdot \sigma^2 \dd s \dd t + e^{rt} \sigma g \cdot b s \dd t + \dd W_t \\
\]

"drift = 0"

\[
\Rightarrow \quad \dd t g + r s \dd s g + s \dd A g + \frac{1}{2} b^2 s^2 \dd s g = \dd W_g
\]

Pricing PDE for Asian option

+ Terminal condition

\[
g(t, s, A) = \varphi \left( \frac{t}{A} \right)
\]

\[
z = \frac{A}{s}
\]

American options

Payoff \( Y(S_t) \) \( 0 \leq t \leq T \)

redeemable at any time \( t \) ('early exercise feature')

\( u(t, S_t) \) = price of American option
It will satisfy linear complementarity conditions

\[
\begin{align*}
    \begin{cases}
    d_t u + r s d_s u + \frac{1}{2} b^2 s^2 d_{ss} u - ru &> 0 \\
    u(t, s) &\geq \psi(s)
    \end{cases}
\end{align*}
\]

with only one of these inequalities strict at all times i.e. if 'hold value' \( u(t, s) > \psi(s) \)

then

\[
    d_t u + r s d_s u + \frac{1}{2} b^2 s^2 d_{ss} u - ru = 0
\]

For American put

with strike \( K \)

\[
\begin{align*}
    u(t, s) &> \psi(s) \\
    d_t u + &\quad = 0
    \end{align*}
\]

American put \( u \) 

Euro put

early exercise
American put $V_t$

European put $V_t$

American call = European call

Euro call $\Rightarrow$ doesn't make sense to exercise early

(And there are dividends $\Rightarrow$ might need to exercise early)

Lookback options

$$dS_t = rS_t dt + \sigma S_t dW_t \quad \text{under } \mathbb{Q}$$

$$Y_t = \max_{0 \leq n \leq t} S_n$$
Payoff: \( Y_T - S_T \text{ at } t = T \)

\[
\mathbb{E} \left[ \exp \left( -r(T-t) \right) (Y_T - S_T) | S_t, Y_t \right]
\]

Apply IFS lemma, make \((-)dt = 0\) \(\Rightarrow\) get PDE

\[ ds, dy \]

1) \( dy_t dY_t = 0 \) because its brach is increasing
\(\Rightarrow\) it has finite 1st variation
\(\Rightarrow\) its quadratic variation is 0

2) \( dy_t \neq 0 dW_t \) \(\Rightarrow\) otherwise \( Y_t \) would be accumulating to 0 quadratic variation

3) \( dy_t dS_t = 0 \)

\[
\begin{align*}
\frac{dg}{dS} + rS \frac{dg}{dt} + \frac{1}{2} \sigma^2 S^2 \frac{d^2 g}{dS^2} &= rg \\
\end{align*}
\]

\[ g(t, 0, Y) = e^{-r(t-t)} Y \]

\[ g(t, Y, Y) = 0 \]

\[ g(t, S, Y) = Y - S \]

\[ \beta(t) \]

\[ \beta(0) = 0, \beta(1) = e \]
\begin{align*}
\beta_0 & = 1 \\
\beta_1 & = e \\
\beta_2 & < 1
\end{align*}