More on random walks

Gambler's ruin problem: what is the prob. that a gambler was $N$ before reaching 0? (i.e., he/she can't lose everything and then win $N$)

$X_n$ a Markov chain, $\{0, 1, \ldots, N\}$ - state space of $X_n$

Transition probabilities: $P_{00} = 1$, $P_{NN} = 1$

$p_{i,j} = \begin{cases} 0 & \text{if } i = j \\ p \text{ for } 0 < p < 1, \quad p \neq \frac{1}{2} \\ 1 - p \text{ for } p = \frac{1}{2}, i = 0, j = 0 \text{ or } p = \frac{1}{2}, i = N, j = N \end{cases}$

Classes:
$\{0\}, \{N\}, \{1, \ldots, N-1\}$

No recurrent transient

Our goal: find $p_i$ - probability that starting at $i$, MC reaches state $N$

What we know: $P_0 = 0$, $P_N = 1$

Idea: "relate" $p_i$ to $p_{i-1}$, $p_{i+1}$

$p_i = p \cdot p_{i+1} + (1-p) \cdot p_{i-1} \quad 1 < i < N$

$p_i = \frac{\mathbb{P}[\text{reaches } N | X_0 = i] = \mathbb{P}[X_1 = i+1 | X_0 = i] \cdot p_{i+1} + \mathbb{P}[X_1 = i-1 | X_0 = i] \cdot p_{i-1}}{\mathbb{P}[X_1 = i+1 | X_0 = i]}$

$p_i = \frac{\mathbb{P}[\text{reaches } N | X_1 = i+1]}{\mathbb{P}[X_1 = i+1 | X_0 = i]}$ $p_i = \frac{\mathbb{P}[\text{reaches } N | X_1 = i]}{\mathbb{P}[X_1 = i | X_0 = i]}$

I used formulas:

$\mathbb{P}(A) = \mathbb{P}(B) \cdot \mathbb{P}(A | B) + \mathbb{P}(B^C) \cdot \mathbb{P}(A | B^C)$

$A = \{ \text{reaches } N \}, \quad B = \{ X_1 = i+1 \}$

$B^C = \{ X_1 = i-1 \}$

Recall

$p \cdot p_i + (1-p) \cdot p_i = p \cdot p_{i+1} + (1-p) \cdot p_{i-1}$

\[ \ldots \]
\[ p(P_{i+1} - P_i) = (1-p)(P_i - P_{i-1}) \]

need to work with these differences

\[ P_{i+1} - P_i = \frac{1-p}{p} (P_i - P_{i-1}) \quad 0 < i < N \]

Add these equations ( \( P_0 = 0 \) - use ) Idea: \( (P_{i+1} - P_i) + (P_i - P_{i-1}) = P_{i+1} - P_i \)

\[ P_i - P_1 = \left( \frac{1-p}{p} + \left( \frac{1-p}{p} \right)^2 + \cdots + \left( \frac{1-p}{p} \right)^i \right) P_1 \]

\[ \Rightarrow \text{ use formula for the sum of geometric progression:} \]

\[ P_i = \frac{\left( 1 - \left( \frac{1-p}{p} \right)^i \right)}{1 - \frac{1-p}{p}} P_1 \]

\[ \left[ \begin{array}{c}
a + a^2 + \cdots + a^n = \frac{a(1 - a^n)}{1 - a} \\
\text{where:} \quad p = \frac{1}{2} ?
\end{array} \right. \]

In particular,

\[ 1 = P_N = \frac{1 - \left( \frac{1-p}{p} \right)^i}{1 - \frac{1-p}{p}} P_1 \Rightarrow \text{can find } P_i \]

Therefore,

\[ P_i = \frac{1 - \left( \frac{1-p}{p} \right)^i}{1 - \left( \frac{1-p}{p} \right)^N} \]

Suppose \( N = \infty \)

\[ P_i = \begin{cases} 1 - \left( \frac{1-p}{p} \right)^i, & p > \frac{1}{2} \\
0, & p < \frac{1}{2} \end{cases} \]

Obvious if \( p < \frac{1}{2} \) ( indeed, prob. of winning < prob. of losing)

Randomizing initial state

So far, we were assuming that \( X_0 \) is fixed.

Let's pick \( X_0 \) randomly:

\[ \xi_i = \mathbb{P}[X_0 = i], \quad 0 \leq \xi_i \leq 1 \]

\[ \sum_i \xi_i = 1 \]

Now quantities like \( \mathbb{P}[X_n = j] \) make sense ( previously we had...
\[ P[X_n = j] = \sum_i \alpha_i \cdot P[X_n = i | X_o = i] \]

\[ = \sum_i P(i) \cdot \alpha_i \]