\( X_n \) is a Markov chain.

\[ i \text{ - some state} \]

If \( i \) is recurrent, then \( X_n \) spends infinite amount of time in \( i \).

If \( i \) is transient, can we find mean time that MC spends \( \geq i \)?

Suppose \( X_n \) is a MC with finite state space \([0,1,\ldots,N]\)

\([0,1,\ldots,M] \) are transient states, \( M < N \)

\( \uparrow \text{why can't put } =? \)

\( \text{(because a MC with finite state space always has a recurrent state)} \)

Let's fix \( i,j \).

\[ S_{ij} = \mathbb{E} \left[ \text{number of time steps when } X_n \text{ visits state } j \mid X_0 = i \right] \]

\( \text{conditional expectation} \)

Condition on the initial transition:

\( (*) \quad S_{ij} = \delta_{ij} + \sum_{k=0}^{M} \mathbb{P}(i,k) S_{kj} \)

\[ \delta_{ij} = \begin{cases} 1 , & i = j \\ 0 , & i \neq j \end{cases} \text{ Dirac delta} \]

\( S_{kj} = 0 \text{ if } k \text{ is a recurrent state, } \)

MC can't jump from a recurrent state to a transient state, \( e.g. \ j \)

Let's re-write \( (*) \) in matrix form:

\[ S = (s_{ij})_{i,j=1}^M \]

\( \mathbb{P}_T = (p_{ij})_{i,j=1}^M \)

\[ S = I + \mathbb{P}_T S \]
\[(I - \rho_T) S = I\]

Therefore,

\[S = (I - \rho_T)^{-1}\]

**Derivative Pricing**

**Asset price model (using random walks)**

Let \(X_n\) be a random walk on \(\mathcal{B}\)

```
1       2
(0) --> (1) --> (2)
```

Another ("dynamise") point of view: start at 0

```
3
(0) <-- (1) <-- (2)
```

\[n = 0, n = 1, n = 2\]

(random walk)

(binomial tree representation)

**Price of stock:**

\[S_n = S(X_n) := \delta^n S_0, \quad S_0 > 0 \quad \text{is fixed} \]

\[\delta S_0 = S_1 n \quad \delta S_0 > 1\]

\[S_0 \left\{\begin{array}{l}
\delta^{-1} S_0 = S_{1d} \\
\delta^{-2} S_0
\end{array}\right.\]

\(S_n\) are r.v. taking values in \((0, \infty)\)

(i.e., \(n\) is a stochastic process)

\(S\) is observable. \(X_n\) is not ("market uncertainty")
$S_n$ is observable, $X_n$ is not ("market uncertainty")

European put option:

"European put option on $S$ with strike $K$ and expiry $T=1$" is a contract that pays at time step 1

$$P(S) = \begin{cases} 0, & \text{if } S \geq K \\ K-S, & \text{if } S < K \end{cases}$$

This is a protection from the risk that at time $t=1$ (future) stock price will fall below $K$.

A client can purchase this contract from a bank. Now the bank has to worry about the risk that the price will fall below $K$.

There is a way to eliminate the risk.

**Replication problem:** How to replicate the option by trading in stock and money markets?

**Ex:** $S_0 = 4$, factor $S = 2$

$$S = \begin{cases} 4, & 2 \cdot 4 = 8 \\ 2^{-1} \cdot 4 = 2 \end{cases}$$

$n=0$ $n=1$

Take put option with $K=S$, $T=1$

$$0 = P_{1,0}$$

$$P(S) = \begin{cases} 0, & \text{if } S \geq K \\ K-S, & \text{if } S < K \end{cases}$$

$$3 = S-2 = P_{1,1}$$

$n=0$ $n=1$
Goal: construct a portfolio at \( n = 0 \) consisting of

\[ \Delta \text{ shares} \]

\[ Y_0 - \Delta S_0 \text{ in a bank account} \]

where \( Y_0 = \text{cost of portfolio} \)

(i.e., initial wealth)

Want this portfolio to replicate the option.

---

**Quiz**

Gautler's ruin problem: \( N = 3 \)

\[
\begin{array}{c}
1 \\
\circlearrowleft \\
0 \\
\end{array}
\quad\quad
\begin{array}{c}
1 \\
\circlearrowleft \\
1 - p \\
\end{array}
\quad\quad
\begin{array}{c}
1 \\
\circlearrowleft \\
1 - p \\
\end{array}
\quad\quad
\begin{array}{c}
1 \\
\circlearrowleft \\
1 - p \\
\end{array}
\quad\quad
\begin{array}{c}
1 \\
\circlearrowleft \\
1 - p \\
\end{array}
\]

\[
P_0 = \text{prob. that this MC will reach state } 3.
\]

Find \( P_1 = \text{prob. that this MC will reach state } 3 \)

(starting at 1)

Let's find what our portfolio \( \Delta \) shares, \( Y_0 - \Delta S_0 \text{ dollars} \)

\[
n = 0: \quad \Delta: S_0 + Y_0 - \Delta S_0 = Y_0
\]

\[
n = 1: \quad \Delta: S_1 + Y_0 - \Delta: S_0 = Y_1
\]

\( S_1 \) is a r.v. taking values \( S_{1,m} = 8 \) or \( S_{1,d} = 2 \)

So, \( Y_1 \) is also a r.v.

\[
Y_{1,m} = \Delta: 8 + Y_0 - \Delta: 4 = 4 \Delta + Y_0
\]
\[
Y_{1,d} = \Delta \cdot 2 + Y_o - \Delta \cdot 4 = -2\Delta + Y_o
\]

Replicate: \[
\begin{align*}
4\Delta + Y_o &= 0 \quad (\text{i.e. } Y_{1,u} = P_{1,u}) \\
-2\Delta + Y_o &= 3 \\
6\Delta &= -3 \
\Delta &= -\frac{1}{2} \quad (\text{i.e. we sell stocks})
\end{align*}
\]

\[
Y_o = -9 \left(\frac{1}{2}\right) = -\frac{9}{2}
\]

It costs \( Y_o = 2 \) to set up the replicating portfolio (this is how much the bank will charge its client for this option).

![Diagram](image)

*Can treat as new option*

**Note:** Probabilities don’t show up.

Instead of option, consider general derivative security

\[
\begin{cases}
V_{1,u} \\
V_{1,d}
\end{cases}
\]

some numbers

Want to create replicating portfolio

We have:

\[
Y_1 = \Delta S_1 + (Y_o - \Delta S_0)
\]

\[
= Y_o + \Delta (S_1 - S_0)
\]

Want to replicate at \( n=1 \):

\[
\begin{cases}
Y_o + \Delta (S_{1,u} - S_0) = V_{1,u} \\
Y_o + \Delta (S_{1,d} - S_0) = V_{1,d}
\end{cases}
\]
where \( S_{1,m} = 6S_0 \), \( S_{1,d} = \delta^{-1}S_0 \).

**Theorem:** We have

\[
V_0 = \tilde{p} V_{1,n} + (1-\tilde{p}) V_{1,d} \quad \ominus
\]

\[
\tilde{p} = \frac{1-\delta^{-1}}{\delta-\delta^{-1}} \quad 1-\tilde{p} = \frac{\delta-1}{\delta-\delta^{-1}}
\]

'Artificial' probabilities

Note: real-world probabilities \( p, 1-p \) don't show up.

\[
\otimes \quad \mathbb{E}_{\tilde{P}} [ V_1 ]
\]

\( V_1 \) is a random variable

\[
\tilde{p} \quad V_{1,n}
\]

\( 1-\tilde{p} \quad V_{1,d}
\]

Can compute this quantity using Prob. Theory / Statistics (e.g. use Monte-Carlo simulations).

\( \tilde{p}, 1-\tilde{p} \) are completely characterized by identity:

\[
S_0 = \mathbb{E}_{\tilde{P}} [ S_1 ] \left( \tilde{p} S_{1,n} + (1-\tilde{p}) S_{1,d} \right)
\]

In other words, process \( S_n \) because we can choose is a martingale with respect to \( \tilde{P} \),

\[
V_{1,n} = S_{1,n} \quad V_{1,d} = S_{1,d}
\]