Stochastic Processes

**Def.** A discrete-time stochastic process is a collection of random variables \( X_n, \ n = 0, 1, 2, \ldots \)

\( n \) is the time (or, rather, time step)

**Def.** A continuous-time stochastic process is a collection of random variables \( X_t, \ t \in (0, +\infty) \)

It is assumed that r.v. \( X_n \) (or \( X_t \)) are taking values in the same space (called “state space”). WLOG, state space will be a subset of \( \mathbb{R} \).

**Def.** For fixed \( n \) (or \( t \)) r.v. \( X_n \) (\( X_t \), respectively) is called the section of process \( X = X_n \) (\( X = X_t \))

Recall: \( X_t = X_t(w) \), \( w \in \Omega \)

We omit \( w \) from now on.

**Def.** Fix \( w \in \Omega \). Then \( n \to X_n(w) \) is a sequence of numbers, \( t \to X_t(w) \) is a function.

In both cases, it is called a trajectory of process \( X \)

**Ex.** (Random walk)

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Experiment: flip n coin => get $1 if H

flip coin n times, \( X_n = \) how much cash we have

provided that \( X_0 = 0 \)

\( X_n \) "cash process" is an example of a stock process

State space? \( \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\} \)
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Given a stochastic process, one can associate with it a

Random walk on $\mathbb{Z}$: elementary but fundamental model
(can model dynamics of an electron or a stock price)

P \left[ X_{n+1} = m-1 \mid X_n = m \right] = \frac{1}{2}

P \left[ X_{n+1} = m+1 \mid X_n = m \right] = \frac{1}{2}

\begin{align*}
X_n \text{ take values in } \mathbb{Z} \\
\text{let's define r.v. } X_n : \\
\text{Formal definition of } X_n \\
\begin{cases}
  \text{if } X_n = m \text{ (dollars)} \\
  \quad X_{n+1} = \begin{cases}
    m-1, \text{ prob. } \frac{1}{2} & (\text{if coin came } T) \\
    m+1, \text{ prob. } \frac{1}{2} & (\text{if coin came } H)
  \end{cases}
\end{cases}
\end{align*}

A compact notation:

A trajectory of $X_n$ (for $n=0,1,2,3$)

"Brownian motion"
Given a stochastic process, one can associate with it a number of random variables:

**Ex. (First passage time of a random walk)**

\( X_n \) stock price (actually, this is a random walk)

\[
\begin{align*}
\text{What's the probability that } X_n \text{ will hit } b \in \mathbb{Z}, \ b > 0, \\
\text{for } n \leq N. \\
\bar{b} = 2, \ N = 30 \text{ (days)}
\end{align*}
\]

**Possible question**

\[
\text{Take } b = 2, \quad \bar{b} = 2
\]

\[
\text{Hit the barrier}
\]

\[
\text{Hitting time (first passage time)}
\]

**First passage time (the first time in random walk } X_n \text{ is equal to } b = 2 \)**

\[
\tau = \min \{ n : X_n = 2 \}
\]

Let's find the distribution of \( \tau \):

\[
\begin{align*}
\text{Distribution of r.v. } \tau \\
\tau = 0 & \quad \mathbb{P}(\tau = 0) = 0 \\
\tau = 1 & \quad \mathbb{P}(\tau = 1) = 0 \\
\tau = 2 & \quad \mathbb{P}(\tau = 2) = \mathbb{P}(\text{win twice in a row}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \\
\tau = 3 & \quad \mathbb{P}(\tau = 3) = 0 \\
& \vdots
\end{align*}
\]

**HW:** find \( \mathbb{P}(\tau = 5) \)

**Def:** A process \( \tilde{X} \) is called a Markov chain if

\[
\mathbb{P}(X_{n+1} = j \mid X_n = i, X_{n-1} = k, \ldots, X_0 = \ell) = \mathbb{P}(X_{n+1} = j \mid X_n = i) =: P_{ij}
\]

In other words, the conditional distribution of \( X_{n+1} \) only depends on \( X_n \) and not on data \( X_{n-1}, \ldots, X_0 \).
given its present state \( X_n \) and past states \( X_{n-1}, X_{n-2}, \ldots, X_0 \) depends only on the present state.

(Informally, \( X_n \) "doesn't have memory")

**Ex:** Random walk is a Markov chain \(<\) check

**HW:** Give example of a process that is not a Markov chain.