Markov chains (cont'd)

\[ X_n \]  discrete time stochastic process

State space \( S \) is a subset of \( \mathbb{Z} \), e.g.

\[ S = \mathbb{Z}, \quad S = \{0, 1, 2, \ldots \}, \quad S = \{0, 1, \ldots, N\} \]

Recall: \( X_n \) is called a Markov chain if the probability that this process will transfer to state \( j \) given that it is currently at state \( i \) depends only on \( i \), i.e.

\[ P[X_{n+1} = j \mid X_n = i] = P[X_{n+1} = j \mid X_n = i, X_m = k, \quad X_0 = \ell] \]

**Def.**: \( P_{ij} = P[X_{n+1} = j \mid X_n = i] \)

are called transition probabilities.

\[ 0 \leq P_{ij} \leq 1, \quad \sum_j P_{ij} = 1 \]

Graph representation of this MC

\[ \begin{align*}
\text{Ex.: (Random walk)} & \\
\text{Suppose: the chance of rain tomorrow depends on whether it is raining today or not} & \\
\text{State of weather: } & 0 \text{ - rain} \\
& 1 \text{ - no rain}
\end{align*} \]
$X_{n} \in \{0, 1\}$ - weather on day $n$

$p_{00} = P[X_{n+1} = 0 | X_{n} = 0] = 0.7$

$p_{01} = 0.3$

$p_{10} + p_{11} = 1$

$p_{10} = P[X_{n+1} = 0 | X_{n} = 1] = 0.4$

$p_{11} = 0.6$

$P = \begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix} = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$

This matrix is called transition probabilities matrix.

Note: the sum of elements in each row is always $1$

![Transition Probabilities Diagram]

**Chapman-Kolmogorov Equations**

Define $n$-step transition probability $p_{ij}^{(n)}$ as the prob. that the Markov chain, currently in state $i$, will transition to state $j$ in $n$ time steps from now.

$p_{ij}^{(n)} := P[X_{k+n} = j | X_k = i]$

Note: $p_{ij}^{(n)}$ (and $p_{ij}$) do not depend on $k$, i.e. transition probabilities do not change with time (follows from the Def. of MC)

Of course, $p_{ij}^{(1)} = p_{ij}$

How to find $p_{ij}^{(n)}$?
Chapman - Kolmogorov equation:

\[ p_{ij}^{(m)} = \sum_k p_{ik}^{(n)} p_{kj}^{(n)} \]

Example:

\[ p_{ij}^{(12)} = \sum_k p_{ik}^{(12)} p_{kj}^{(12)} \]
\[ p_{ij}^{(13)} = \sum_k p_{ik}^{(13)} p_{kj}^{(13)} \]

\[ p_{ik}^{(n)} p_{kj}^{(n)} = \text{prob. MC will transition from } i \text{ to } k \text{ in } n \text{ time steps, and then will transition from } k \text{ to } j \text{ in } n \text{ time steps} \]

\[ \Rightarrow \text{we need to sum over all intermediate states } k \]

Proof:

Assume \( X_0 = i \)

\[ p_{ij}^{(n+n)} = \mathbb{P}[X_{n+n} = j] = \sum_k \mathbb{P}(X_{n+n} = j, X_n = k) \]

\[ = \sum_k \left[ \mathcal{B}_n = \{ X_n = k \} \right] \mathbb{P}(A) = \sum \mathbb{P}(A \cap \mathcal{B}_n) \]

formula for conditional prob. \( A = \{ X_{n+m} = j \} \) \( \mathcal{B}_n = \{ X_n = k \} \)

\[ \sum_k \mathbb{P}(X_{n+m} = j | X_n = k) \cdot \mathbb{P}(X_n = k) = \sum_k p_{kj}^{(n)} p_{ik}^{(n)} \]

Denote:

\[ p^{(n)} = (p_{ij}^{(n)}) \] The matrix of \( n \) step transition prob.

Proposition (Equivalent form of C-K equations):
Corollary: \[ P^n = P \cdot P \cdot P \cdots P \]

where \( P = (p_{ij}) \)

Ex. (Weather forecast)

\[ P = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} \]

Suppose it is raining today. What is the probability of rain in 2 days from now? i.e. find \( P^{(2)} \)

Compute \( P^2 = P^{(2)} \)

\[ P = \begin{bmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{bmatrix} \]

Actually, \( P = \begin{bmatrix} 0.7 & 0.3 \\ 0.5 & 0.5 \end{bmatrix} \)

Can you guess \( \lim_{n \to \infty} P^n \)? Common sense: \( \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \)

Ex. (Mood swings)

- Cheerful: C
- So-so: S
- Glum: G

(Mood changes day to day)

\[ \frac{1}{4} \quad C \quad \quad \frac{1}{2} \quad S \quad \frac{3}{4} \quad G \]

The sum of probabilities at arrows coming from each state must be equal to 1

What is the prob. of G if he/she is C today?

0: G in 2 days

\[ \begin{bmatrix} 1/4 & 0 & 3/4 \end{bmatrix} \begin{bmatrix} 0 & 0.7 \end{bmatrix} = \begin{bmatrix} 0 & 0.7 \end{bmatrix} \]
Find $P(3) = \frac{1}{8} P^2 = \begin{bmatrix} \frac{1}{4} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{3}{4} \end{bmatrix}$

Classification of States

**Def.** State $j$ is said to be accessible from state $i$ if $P^{(n)}_{ij} > 0$ for some $n > 0$ $j \rightarrow i$

In particular, every state is accessible from itself.

**Def.** Two states $i, j$ that are accessible from each other are said to communicate.

Each state is communicating with itself.

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  S
 / \
C  \  G
 /    \
C    S
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C ↔ C  C ↔ S
S ↔ S  G ↔ S
G ↔ C

**Def.** Two states that communicate are said to be in one class.

**Def.** A Markov chain is said to be irreducible if there is only one class, i.e., all states communicate.

In the example above, state $G$ (even though it is accessible from $S$ and $C$) doesn’t communicate with $S$ and $C$. 
There are two classes: \( \{S, C\}, \{G\} \)