Review of Probability

The basic model of random experiment

Def: A finite probability space is a pair \((\Omega, P)\)

1) Sample space \(\Omega\) ("all possible outcomes of the experiment")

2) Probability measure \(P\) that assigns to each element of \(w \in \Omega\) a number \(P(w)\) between 0 and 1 ("outcome"") such that \(0 \leq P(w) \leq 1\) ("probability of outcome \(w\)"")

\[ \sum_{w \in \Omega} P(w) = 1 \]

If \(P(w) = 0\) \(\Rightarrow\) this outcome doesn't happen

\(P(w) = 1\) \(\Rightarrow\) \(w\) will happen for sure

Ex: Coin toss space

\[ \Omega = \{ H, T \} \]

\(P(H) = \frac{1}{2}\) \(\quad P(T) = \frac{1}{2}\)

Ex: Toss a coin 10,000 times \(\Rightarrow\) 50% head, 50% tail

Can ask non-trivial questions: if one of the boxes got \(H\), what's the probs. of getting \(T\)?
**Def.** An event $A$ is a collection of outcomes, i.e. a subset of $\Omega$.

**Ex.** $A = \text{get } T \text{ in at least one of tosses (toss 2 times)}$

$A = \{HN, TH, TT\}$ clearly, a subset of $\Omega$

**Def.** Probability of an event $A$ is defined to be

$$P(A) = \sum_{\omega \in A} P(\omega)$$

sometimes “outcome” $\omega$ is called “elementary event.”

**Ex.** Toss coin 2 times

$A = \text{get } H \text{ at the second toss } \implies P(A) = ?$

$A = \{HN, TH\}$

$$P(A) = P(HN) + P(TH) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Can an event have probability 0?

**Ex.** $\Omega = \{H,T,E\}$

$P(H) = \frac{1}{2}$

$P(T) = \frac{1}{2}$

$P(E) = 0 \implies A = \{E\}$ has probability 0.

**Properties of $P$:**

1. If events $A$ and $B$ are mutually exclusive, i.e. $A \cap B = \emptyset$ (denote then $AB = A \cap B$)

$$P(A \cup B) = P(A) + P(B)$$

2. For any two events $A$, $B$

$$P(A \cup B) = P(A) + P(B) - P(AB)$$
Ex: Toss coin twice

\[ A = \{ \text{HH, HT} \} , \quad B = \{ \text{HH, TH} \} \]

\[ P(A \cup B) = P(A) + P(B) - P(AB) \]

\[ P(A \cup B) = P(\text{HH}) + P(\text{TH}) - P(\text{HH, TH}) \]

\[ P(AB) = P(\text{HH}) = \frac{1}{4} \]

\[ P(A \cup B) = \frac{1}{2} + \frac{1}{4} - \frac{1}{4} = \frac{3}{4} \]

Directly: \[ P(A \cup B) = P(\{ \text{HH, HT, TH} \}) = \frac{3}{4} \]

Random variables, distributions and expectations

Def: Let \((\Omega, \mathcal{F}, P)\) be a finite prob. space

A random variable is a real-valued function on \(\Omega\).