Brownian motion

Itô integral  (know how to compute $\int_0^t w_s^2 dw_s$

SDE $\int_0^t (w_s^2 + 5) dw_s = \int_0^t w_s^2 dw_s + \int_0^t 5 dw_s$

Ex: GBM

$$dS_t = \alpha S_t dt + \sigma S_t dw_t, \quad S_0 = c > 0$$

$$dW_t + (\alpha - \frac{1}{2} \sigma^2) dt = \sigma dW_t$$

$$S_t = Ce^t$$

- sort of a Brownian motion that is always positive

Intuitively: $S_t$ is small $\Rightarrow$ "$dS_t$" is small

Ex: Mean-reverting process $X_t$

$$dX_t = (1 - X_t) dt + \sigma dw_t$$

When $X_t > 1 \Rightarrow$ "$dX_t$ is negative"

$X_t < 1 \Rightarrow$ "$dX_t$ is positive"

$\Rightarrow X_t$ will be mean-reverting to 1.

This process can be used to model interest rates.

There is an explicit formula for $X_t$ (HW #4)

Mating times, maximum of BM

Let $a > 0$
Let $a > 0$

$T_a$ is called hitting time.

$T_a$ is a random variable = time when BM hits barrier $a$ for the first time.

BM will hit barrier $a$ at some time with probability 1, so we may assume that $T_a$ is finite.

Precise definition: $T_a = \inf \{ t : W_t > a \}$

Ex: Suppose $W_t$ is the cash process of a gambler. Then $T_a$ is the first time he/she wins $\$a$.

Typical question: find $P[T_a \leq 6]$

6 a.m.

We need to know the distribution of $T_a$:

$$P[W_t > a] = P[U_t > a | T_a \leq t] P[T_a \leq t]$$

$$+ P[U_t > a | T_a > t] P[T_a > t]$$

Claim: $P[W_t \geq a | T_a \leq t] = \frac{1}{2}$

W.t < a for all s < t means that $W_t \geq a$ couldn't happen

Claim: $P[W_t \geq a | T_a \leq t] = \frac{1}{2}$

For each possible contraction, we can reflect

because $P[W_t \geq a | T_a \leq t] = P(U_t \leq a | T_a \leq t)$

and, on the other hand,

$$P[W_t \geq a | T_a \leq t] = 1 - P(U_t \leq a | T_a \leq t)$$

Therefore, from $(*)$

$$P[W_t \geq a] = \frac{1}{2} P[T_a \leq t]$$
\[ \mathbb{P}[W_t > a] = \frac{1}{2} \mathbb{P}[T_a \leq t] \]

\[ \Rightarrow \mathbb{P}[T_a \leq t] = 2 \cdot \frac{1}{(2\pi)^{1/2} t} \int_0^\infty e^{-\frac{x^2}{2t}} \, dx \quad - \text{distribution of hitting time } T_a \]

\[ m_t = \max_{s \in [0,t]} W_s \]

Distribution:

\[ \mathbb{P}[m_t > c] = \mathbb{P}[T_c \leq t] \]

\[ = 2 \mathbb{P}[W_t > c] = \frac{2}{(2\pi)^{1/2} t} \int_c^\infty e^{-\frac{x^2}{2t}} \, dx \]