FELLER EVOLUTION FAMILIES AND PARABOLIC EQUATIONS WITH FORM-BOUNDED VECTOR FIELDS

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Abstract. We show that the weak solutions of parabolic equation $\partial_t u - \Delta u + b(t, x) \cdot \nabla u = 0$ with vector field $b(t, x)$ satisfying form-boundedness condition constitute a Feller evolution family and, thus, determine a strong Markov process. Our proof uses a Moser-type iterative procedure and an a priori estimate on the $L^p$-norm of the gradient of solution in terms of the $L^q$-norm of the gradient of initial function.

1. Introduction and results

1.1. Introduction. Consider Cauchy problem ($s \geq 0$)

$$\begin{align*}
  (\partial_t - \Delta - b(t, x) \cdot \nabla) u &= 0, & (t, x) \in (s, \infty) \times \mathbb{R}^d, \\
  u(+s, s, \cdot) &= f(\cdot),
\end{align*}$$

where $d \geq 3$, $b \in L^1_{\text{loc}}([0, \infty) \times \mathbb{R}^d, \mathbb{R}^d)$, $f \in L^2_{\text{loc}}(\mathbb{R}^d)$.

We study the following question: if one restricts the choice of initial function $f$ to space $C_\infty(\mathbb{R}^d) := \{f \in C(\mathbb{R}^d) : \lim_{x \to \infty} f(x) = 0\}$ (endowed with sup-norm $\| \cdot \|_\infty$), under what assumptions on the vector field $b$ the (unique) weak solution of (1), (2) if given by a Feller (Feller-Dynkin) evolution family? That is, there exists a family of bounded linear operators $U = (U(t, s))_{0 \leq s \leq t < \infty} \subset L(C_\infty(\mathbb{R}^d))$ such that

\begin{itemize}
  \item[(E1)] $U(s, s) = \text{Id}$, $U(t, s) = U(t, r)U(r, s)$ for all $0 \leq s \leq r \leq t$,
  \item[(E2)] mapping $(t, s) \mapsto U(t, s)$ is strongly continuous in $C_\infty(\mathbb{R}^d)$,
  \item[(E3)] operators $U(t, s)$ are positivity-preserving and $L^\infty$-contractive:
    \[ U(t, s)f \geq 0 \quad \text{if} \quad f \geq 0, \quad \text{and} \quad \|U(t, s)f\|_\infty \leq \|f\|_\infty, \quad 0 \leq s \leq t, \]
  \item[(E4)] function $u(t, s, \cdot) := U(t, s)f$ $(f \in C_\infty(\mathbb{R}^d))$ is a weak solution of equation (1).
\end{itemize}

Using the Riesz-Markov representation theorem [EG, Ch.1], it is not difficult to show that for every $T > 0$ operators $V(\tau, \sigma) := U(T - \tau, T - \sigma)$ $(0 \leq \tau \leq \sigma \leq T)$ determine (sub-)probability transition functions $p(\tau, x; \sigma, dy)$. By [GvC] Theorem 2.22, these are the transition functions of
a strong Markov process \( X_t \) (\( 0 \leq t \leq T \)) that is quasi-left-continuous and has right continuous trajectories with left limits, and

\[
V(\tau, \sigma)f = \int_{\mathbb{R}^d} p(\tau, x; \sigma, dy)f(y) = E_{\tau,x}[f(X_{\sigma})], \quad f \in C_{\infty}(\mathbb{R}^d).
\]

The relationship between process \( X_t \) and the formal differential expression in (1) is given by (E4).

The problem of constructing and investigating the processes associated with parabolic differential operators having singular coefficients attracted the interest of many researchers, see [BKR, BG, GvC] and references therein. The present work contributes to this study. We are dealing with non-symmetric and non-stationary setup, where many standard techniques are no longer available. We consider the class of form-bounded vector fields \( b(t, x) \)\(^1\) This class contains \( L^d \), weak \( L^d \) and Kato class \( K_d \).

1.2. Main result.

**Definition 1.** The vector field \( b(\cdot, \cdot) \) is said to belong to class \( (F_\beta) \) if \( b \in L^{2}_{loc}([0, \infty) \times \mathbb{R}^d, \mathbb{R}^d) \) and there exists \( \beta < \infty \) such that

\[
\int_0^\infty \| b(t, \cdot)\varphi(t, \cdot) \|^2_2 dt \leq \beta \int_0^\infty \| \nabla \varphi(t, \cdot) \|^2_2 dt + \int_0^\infty g(t) \| \varphi(t, \cdot) \|^2_2 dt
\]

for some \( g = g_\beta \in L^1_{loc}([0, \infty)) \), for all \( \varphi \in C^\infty_0([0, \infty) \times \mathbb{R}^d) \). Here \( \| \cdot \|_2 \) is the norm in \( L^2(\mathbb{R}^d) \).

**Example 1.** (1) \( L^d(\mathbb{R}^d) \subset \bigcap_{\beta > 0} (F_\beta) \).

(2) \( b(x) = x|x|^{-2} \) belongs to \( (F_\beta) \) with \( \beta = (2/(d - 2))^2 \), \( g \equiv 0 \) (Hardy inequality).

(3) If \( |b| \in L^{d,\infty}(\mathbb{R}^d) + L^\infty(\mathbb{R}^d) \), then \( b = b(x) \) is in \( (F_\beta) \) with \( \beta \) depending on the weak norm of \( |b| \).

(4) More generally, any vector field \( b(t, x) \) such that for some \( c_1, c_2 > 0 \)

\[
|b(t, x)|^2 \leq c_1 |x - x_0|^{-2} + c_2 |t - t_0|^{-1} (\log(e + |t - t_0|^{-1}))^{-1-\varepsilon}, \quad \varepsilon > 0, \quad (t, x) \in [0, \infty) \times \mathbb{R}^d,
\]

belongs to class \( (F_\beta) \) with \( \beta = c_1 (2/(d - 2))^2 \).

(5) If \( x = (y, z), y \in \mathbb{R}^n, z \in \mathbb{R}^m, n + m = d \), then \( b(x) = C_1 y|y|^{-2} + C_2 z|z|^{-2} \) is in \( (F_\beta) \) with appropriate \( \beta, g \equiv 0 \).

(6) \( \beta(x) = C(1_{1+\delta} - 1_{1-\delta})e(|x| - 1)^{-1} (e \in \mathbb{R}^d, 0 < \delta < 1) \) is in \( (F_\beta) \) with appropriate \( \beta \), where \( 1_{1+\delta} \) is the characteristic function of the open ball centered at the origin and having radius \( 1 \pm \delta \).

Some other examples are given by improved Hardy inequalities, see [GM].

**Definition 2.** A function \( u \in L_{loc}^2((s, \infty) \times \mathbb{R}^d) \) is said to be a weak solution of equation (1) if \( \nabla u \) (understood in the sense of distributions) is in \( L_{loc}^1((s, \infty) \times \mathbb{R}^d, \mathbb{R}^d) \), \( b \cdot \nabla u \in L_{loc}^1((s, \infty) \times \mathbb{R}^d) \), and

\[
\int_0^\infty \langle u, \partial_t \psi \rangle dt - \int_0^\infty \langle u, \Delta \psi \rangle dt - \int_0^\infty \langle b \cdot \nabla u, \psi \rangle dt = 0 \tag{3}
\]

\(^1\)From the viewpoint of regularity theory of equation (1), this class captures the critical order of singularity of \( b \) both in time and space variables, e.g., there are counterexamples to uniqueness of solution of (1), (2) if \( b \) is replaced by \( cb \) with \( c > 1 \) large, cf. example in Section 2.
for all $\psi \in C_0^\infty((s, \infty) \times \mathbb{R})$, where

$$\langle h, g \rangle = \langle hg \rangle := \int_{\mathbb{R}^d} h(x)g(x)dx$$

is the inner product in $L^2(\mathbb{R}^d)$.

**Definition 3.** A weak solution of (1) is said to be a weak solution to Cauchy problem (1), (2) if

$$\lim_{t \to +s} \langle u(t, s, \cdot) , \xi \rangle = \langle f , \xi \rangle$$

for all $\xi \in L^2(\mathbb{R}^d)$ having compact support.

Denote $\omega_q := \frac{q-1}{\sqrt{d+q-2}}$.

**Theorem 1 (Main result).** Suppose a vector field $b(\cdot, \cdot)$ belongs to class $(F_\beta)$. If

$$\beta < \min \{ \frac{2}{(d-2)^2} , 1 \} \omega_2^2 \max \{ d-2 , 2 \} \quad (< 1), \quad (4)$$

then there exists a Feller evolution family $\{U(t,s)\}_{0 \leq s \leq t} \subset \mathcal{L}(C_\infty(\mathbb{R}^d))$ that produces weak solution to Cauchy problem (1), (2), i.e. $(E1)$--$(E4)$ hold true.

In the stationary case, i.e. if $b = b(x)$ satisfies for some constant $c_\beta$, for all $\phi \in C_0^\infty(\mathbb{R}^d)$

$$\|b\phi\|_2^2 \leq \beta \|\nabla \phi\|_2^2 + c_\beta \|\phi\|_2^2,$$

there exists a positivity preserving contraction $C_0$-semigroup $\{U(t)\}_{t \geq 0}$ on $C_\infty(\mathbb{R}^d)$ such that $u(t, \cdot) := U(t)f$ $(f \in C_\infty(\mathbb{R}^d))$ is a weak solution of Cauchy problem (1), (2) with $b = b(x)$.

Theorem 1 in the stationary case and under the extra assumption $b(\cdot) \in L^2(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ is due to [KS, Theorem 2]. The extra constraint on $b$ in [KS] arises in the verification (in Trotter-Kato-Neveu theorem) that the constructed pseudo-resolvent is a resolvent or, equivalently, that the constructed limit of approximating semigroups is strongly continuous in $C_\infty(\mathbb{R}^d)$. We modify their iterative procedure to automatically yield strong continuity.

**Remark 1.** If $b$ is in class $(K_{d+1})$, then the fundamental solution of (1) admits a Gaussian upper bound [Se]. If $b$ is in $(K_{d+1}) \cap (F_\beta)$ with $\beta$ satisfying (1), then $\{U(t,s)\}_{0 \leq s \leq t}$ can be extended to a (strongly continuous) evolution family in $\mathcal{L}(C_b(\mathbb{R}^d))$, $C_b(\mathbb{R}^d) := \{ f \in C(\mathbb{R}^d) : \sup_x |f(x)| < \infty \}$ is endowed with the sup-norm.

**Remark 2.** In the assumptions of Theorem 1 given $p > (1 - \sqrt{\beta/4})^{-1}$, the formula

$$U_p(t,s) := \left( U(t,s) \big|_{L^p(\mathbb{R}^d) \cap C_\infty(\mathbb{R}^d)} \right)_{L^p(\mathbb{R}^d) \rightarrow L^p(\mathbb{R}^d)}$$

determines a (strongly continuous) evolution family in $\mathcal{L}(L^p(\mathbb{R}^d))$. The proof is obtained from Theorem 1 estimate $\|u(t,s, \cdot)\|_p \leq C_T \|f\|_p$, $0 \leq s \leq t \leq T$ (see [13] below) and the Dominated Convergence Theorem [EG, Ch.1].
1.3. **Proof of Theorem 1**: a parabolic variant of the iterative procedure of Kovalenko-Semenov. The stationary case is immediate, once we construct a Feller evolution family.

We fix $T > 0$, and denote $D_T := \{(s,t) \in \mathbb{R}^2 : 0 \leq s \leq t \leq T\}$.

We will need a regular approximation of $b$: vector fields $\{b_m\}_{m=1}^\infty \subset C_0^\infty([0,\infty) \times \mathbb{R}^d,\mathbb{R}^d)$ that satisfy

\begin{enumerate}[(C_1)]
  \item $b_m \to b$ in $L^2_{\text{loc}}([0,\infty) \times \mathbb{R}^d,\mathbb{R}^d)$,
  \item for all $\varphi \in C_0^\infty([0,\infty) \times \mathbb{R}^d)$
    \[ \int_0^\infty \| b_m(t,\cdot)\varphi(t,\cdot) \|^2 dt \leq \left( \beta + \frac{1}{m} \right) \int_0^\infty \| \nabla \varphi(t,\cdot) \|^2 dt + \int_0^\infty g(t) \| \varphi(t,\cdot) \|^2 dt. \]
\end{enumerate}

(Such $b_m$’s can be constructed by formula $b_m := \eta_m * \mathbf{1}_m b$, where $1_m$ is the characteristic function of set $\{(t,x) \in \mathbb{R} \times \mathbb{R}^d : |b(t,x)| \leq m, |x| \leq m, 0 \leq |t| \leq m\}$, * is the convolution on $\mathbb{R} \times \mathbb{R}^d$, and $\{\eta_m\} \subset C_0^\infty(\mathbb{R} \times \mathbb{R}^d)$ is an appropriate family of mollifiers.)

The construction of the Feller evolution family goes as follows. Let $f \in C_0^\infty(\mathbb{R}^d)$. The unique (classical) solution of Cauchy problem

\[(\partial_t - \Delta - b_m(t,x) \cdot \nabla)u = 0, \quad u(+s,s,\cdot) = f \quad (5)\]

is given by a Feller evolution family $(U_m(t,s))_{0 \leq s \leq t} \subset \mathcal{L}(C_0^\infty(\mathbb{R}^d))$; in particular,

\[ \| U_m f \|_{L^\infty(D_T,C_0^\infty(\mathbb{R}^d))} \leq \| f \|_{\infty} \quad (6) \]

We define

\[ Uf := \lim_{m \to \infty} U_m f \quad \text{in} \quad L^\infty(D_T,C_0^\infty(\mathbb{R}^d)) \quad (7) \]

Assuming that the convergence in (7) has been established, we can use the fact that $C_0^\infty(\mathbb{R}^d)$ is dense in $C_\infty(\mathbb{R}^d)$, so that in view of (6) we can extend $(U(t,s))_{0 \leq s \leq t}$ to a strongly continuous family of bounded linear operators in $\mathcal{L}(C_\infty(\mathbb{R}^d))$, which we denote again by $(U(t,s))_{0 \leq s \leq t}$. That $(U(t,s))_{0 \leq s \leq t}$ is the required Feller evolution family is the content of the following

**Proposition 2.** $(U(t,s))_{0 \leq s \leq t}$ defined by (7) satisfies (E1)-(E4).

The main difficulty is in establishing the convergence in (7). The proof of convergence uses a parabolic variant of the iterative procedure of Kovalenko-Semenov [KS], consisting of three components:

- an a priori estimate (Lemma 3 below),
- an iteration inequality (Lemma 4 below),
- a convergence result in $L^r$ (Lemma 5 below),

The first two components are assembled together in Corollary 5; the latter and the third component yield the convergence in (7).

To shorten notation, set $u_m(t,s,\cdot) = U_m(t,s)f(\cdot)$, $f \in C_0^\infty(\mathbb{R}^d)$.

**Lemma 3** (A priori estimates). Suppose $b$ is in $(F_\beta)$ with $\beta < \frac{4}{q} \omega^2_0$, $q \geq 2$. Then
Lemma 6. Then $r > C\|f\|_q$, where $C = C(q,T) < \infty$ does not depend on $m$.

(2) $\|\nabla u_m(t,s,\cdot)\|_q \leq C_1\|\nabla f\|_q$, where $C_1 = C_1(q,T) < \infty$ does not depend on $(s,t) \in D_T$ and $m$.

Lemma 4 (Iteration inequality). Suppose $b$ satisfies $(F_\beta)$ with $\beta < 4$. Then there exists a $m_0$ such that for all $p \geq p_0 > \frac{2}{2-\sqrt{\beta}}$ and all $m,n \geq m_0$

$$
\|u_m - u_n\|_{L^{\infty}(D_T \times \mathbb{R}^d)} \leq \left( C_0 \left( \beta + \frac{1}{m_0} \right) \|\nabla u_m\|^2_{L^{2\sigma}(D_T \times \mathbb{R}^d)} \right)^{\frac{1}{p}} \left( p^{2k} \right)^{\frac{1}{p}} \|u_m - u_n\|_{L^{\sigma'}(D_T \times \mathbb{R}^d)},
$$

where $1 < \sigma' < \frac{d}{d-2}$, $\sigma > \frac{d}{2}$, $\frac{1}{\sigma} + \frac{1}{\sigma'} = 1$, $k = k(\beta)$, and $C_0 = C_0(T) < \infty$ does not depend on $m$.

Lemmas 3(1) and 4 can be combined to yield

Corollary 5. In the assumptions of Theorem 1, for any $p_0 > \frac{2}{2-\sqrt{\beta}}$ there exist constants $B < \infty$, $\gamma := (1 - \sigma'(d-2)) (1 - \sigma'(d-2) + 2\sigma'/p_0)^{-1} > 0$ independent of $m,n$ such that

$$
\|u_n - u_m\|_{L^{\infty}(D_T \times \mathbb{R}^d)} \leq B\|u_n - u_m\|_{L^{p_0}(D_T \times \mathbb{R}^d)} \quad \text{for all } n,m.
$$

Lemma 6. If $b \in (F_\beta)$ with $\beta < 1$. Then the sequence $\{u_m\}$ from Corollary 5 is fundamental in $L^r(D_T \times \mathbb{R}^d)$, $r \geq 2$.

Now we are in a position to prove convergence in (1). Fix $f \in C^0_\infty(\mathbb{R}^d)$, and choose $r = 2$ in Lemma 6. Then $r > \frac{2}{2-\sqrt{\beta}}$ since $\beta$ is less than 1, and we can take $p_0 := r$ in Corollary 5. Now, Corollary 5 and Lemma 6 imply that the sequence $\{u_m\}$ is fundamental in $L^\infty(D_T, C^\infty(\mathbb{R}^d))$, and hence $(U(t,s))_{0 \leq s \leq t}$ in (7) is well defined. The proof of Theorem 1 is complete.

Remark 3. Note that the extra constraint on $\beta$ (that is, in addition to $\beta < 1$) in Theorem 1 comes solely from Lemma 3(1).

2. Example

If $b$ belongs to $(F_\beta)$ with $\beta < 1$, then a weak solution of Cauchy problem (1), (2) with initial data $f \in C^\infty(\mathbb{R}^d)$ is unique in $L^\infty([0,T], C^\infty(\mathbb{R}^d))$; uniqueness of solution fails if $\beta$ is large: consider equation (1) with

$$
b(t,x) = 2\kappa \alpha (-\ln t)^{\alpha-1} \frac{x}{|x|^2}, \quad 0 < t < 1, \quad x \in \mathbb{R}^d,
$$

where $\alpha \geq 1$, $\kappa > 0$. Then

$$
u(t,x) = (4\pi t)^{-\frac{d}{2}} \exp \left( -\kappa (-\ln t)^\alpha - \frac{|x|^2}{4t} \right)
$$

is a weak solution of (1) in $C^\infty(\mathbb{R}^d)$.

If $\alpha = 1$, $\kappa > \frac{d}{2}$, then $\lim_{t \to 0} \sup_{x \in \mathbb{R}^d} |u(t,x)| = 0$, i.e. there exist multiple solutions to Cauchy problem (1), (2) with initial data $C^\infty(\mathbb{R}^d)$. We note that condition $(F_\beta)$ is satisfied for such $b$ only if $\beta > 4d^2/(d-2)^2$ (using Hardy inequality).

If $\alpha > 1$, $\kappa > 0$, then solution of Cauchy problem for (1), (2) with initial data $C^\infty(\mathbb{R}^d)$ is again not unique: $\lim_{t \to 0} \sup_{x \in \mathbb{R}^d} |u(t,x)| = 0$. Therefore, the singularity of vector field $c(x) \frac{x}{|x|^2}$ can not
be strengthened in the time variable even if \( c > 0 \) is small (condition \((F_{\beta})\) is not satisfied for any \( c \neq 0 \)).

3. PROOFS

Preliminaries. 1. It will be convenient to prove Lemmas 3.6 and Corollary 5 for solutions of Cauchy problem

\[
(\partial_t - \Delta - b_m(t, x \cdot \nabla - h'(t))v = 0, \quad v(+s, s, \cdot) = f, (10)
\]

where a locally absolutely continuous function \( h : [0, \infty) \rightarrow \mathbb{R}, \ h(0) = 0 \), will be the subject of our choice; since the (unique) solution \( v_m \) of (10) is related to the solution \( u_m \) of (5) via identity

\[
v_m(t, s, \cdot) = u_m(t, s, \cdot) e^{h(t) - h(s)}, \quad x \in \mathbb{R}^d, \quad (t, s) \in DT = \{(s, t) \in \mathbb{R}^2 : 0 \leq s \leq t \leq T \} (11)
\]

the assertions of Lemmas 3.6 and Corollary 5 for \( u_m \) will follow.

Also, it is immediate from (6) and (11) that

\[
\|v_m\|_{L^\infty(DT, C^\infty(\mathbb{R}^d))} \leq c\|f\|_{\infty} (12)
\]

for some \( c = c(T, h) < \infty \) independent of \( m \).

2. We will use the following well known result (in fact, valid in a much greater generality). Suppose \( b \) belongs to \((F_{\beta})\) with \( \beta < 1 \). If \( p > \left(1 - \sqrt{\beta/4}\right)^{-1}, f \in L^p(\mathbb{R}^d), \) the (unique) weak solution of (1) in \( L^p([0, T] \times \mathbb{R}^d) \) such that

\[
limit_{t \rightarrow +s} \langle u(t, s, \cdot), \xi \rangle = \langle f, \xi \rangle
\]

for all \( \xi \in L^p(\mathbb{R}^d) \) having compact support, \( \frac{1}{p} + \frac{1}{p'} = 1 \), satisfies

\[
\|u(\tau, s, \cdot)\|_p \leq C\|f\|_p, \quad (s, \tau) \in DT (13)
\]

where \( C = C(T) < \infty \). (See Appendix for the proof of (13).)

3.1. Proof of Lemma 3. Let \( v_m \) be the (classical) solution of Cauchy problem (10) with \( f \in C^\infty_0(\mathbb{R}^d) \). In view of (11), it suffices to prove that for all \( q \geq 2 \)

\[
\|\nabla v_m\|_{L^{q\beta}(DT \times \mathbb{R}^d)} \leq C'\|\nabla f\|_q \quad (\Rightarrow \text{Lemma 3.1}), (14)
\]

where \( C' = C'(q, T) > 0 \) does not depend on \( m \), and

\[
\|\nabla v_m(t, s, \cdot)\|_q \leq \|\nabla f\|_q \quad (\Rightarrow \text{Lemma 3.2}). (15)
\]

In what follows, we omit index \( m \) wherever possible: \( v = v_m \). Denote \( w = \nabla v, w_r = \frac{\partial}{\partial x_r}v, 1 \leq r \leq d \). Let us fix some \( (s, \tau) \in DT, q \geq 2 \). Define

\[
\varphi_r := -\frac{\partial}{\partial x_r} \left(w_r|w|^{q-2}\right), \quad 1 \leq r \leq d,
\]

\[
I_q = \int_s^\tau \left|\sum_{r=1}^d |\nabla w_r|^2\right| \, dt \geq 0, \quad J_q = \int_s^\tau \left|\sum_{r=1}^d \left|\nabla w_r|^2\right|\right| \, dt \geq 0.
\]
We multiply the equation in (10) by \( \varphi_r \), integrate in \( t \) and \( x \), and then sum over \( r \) to get

\[
S := \sum_{r=1}^{d} \int_s^\tau \left< \varphi_r, \frac{\partial v}{\partial t} \right> dt
= \sum_{r=1}^{d} \int_s^\tau \left< \varphi_r, \Delta v \right> dt + \sum_{r=1}^{d} \int_s^\tau \left< \varphi_r, b_m w \right> dt + \sum_{r=1}^{d} \int_s^\tau \left< \varphi_r, h^r(t) v \right> dt =: S_1 + S_2 + S_3.
\]

We can re-write

\[
S = \frac{1}{q} \int_s^\tau \frac{\partial}{\partial t} \langle |w|^q \rangle dt = \frac{1}{q} \left( \langle |w(\tau, s, \cdot)|^q \rangle - \frac{1}{q} \langle |\nabla f(\cdot)|^q \rangle \right)
\]

(the fact that \( w(s, s, \cdot) = \nabla f(\cdot) \) follows by differentiating in \( x_i \), for each \( 1 \leq i \leq d \), the equation in (10) and the initial function \( f \), solving the resulting Cauchy problem, and then integrating its solution in \( x_i \) to see that it is indeed the derivative of \( v \) in \( x_i \)). Further,

\[
S_1 = -\sum_{r=1}^{d} \int_s^\tau \left< \frac{\partial}{\partial x_r} (w_r |w|^{q-2}), \Delta v \right> dt = -\sum_{r=1}^{d} \int_s^\tau \left< \nabla (w_r |w|^{q-2}), \nabla w_r \right> dt
= -\int_s^\tau \langle |w|^{q-2} \sum_{r=1}^{d} |\nabla w_r|^2 \rangle dt - \frac{1}{2} \int_s^\tau \langle |\nabla |w|^{q-2}, |\nabla |w|^2 \rangle dt = -I_q - (q - 2)J_q.
\]

Next,

\[
S_2 = -\int_s^\tau \langle |w|^{q-2} \Delta v, b_m \cdot w \rangle - \int_s^\tau \langle w \cdot \nabla |w|^{q-2}, b_m \cdot w \rangle dt =: F_1 + F_2.
\]

Let us estimate \( F_1 \) and \( F_2 \) as follows. By elementary inequality \( ab \leq \frac{\gamma}{4} a^2 + \frac{1}{\gamma} b^2 \) (\( \gamma > 0 \)), we have

\[
|F_1| \leq \int_s^\tau \langle |w|^{\frac{q-2}{2}} |\Delta v||w|^{\frac{q-2}{2}} |b_m||w| \rangle dt \leq
\frac{\gamma}{4} \int_s^\tau \langle |w|^{q-2} |\Delta v|^2 \rangle dt + \frac{1}{\gamma} \int_s^\tau \left< \left( |b_m||w|^{\frac{q}{2}} \right)^2 \right> dt \quad \text{(by (C2))}
\]

\[
\leq \frac{\gamma}{4} \int_s^\tau \langle |w|^{q-2} |\Delta v|^2 \rangle dt + \frac{1}{\gamma} \left[ \beta + 1/m \right] \frac{q^2}{4} J_q + \int_s^\tau g(t) \langle |w|^q \rangle \leq
\beta + 1/m \right] \frac{q^2}{4} J_q + \int_s^\tau g(t) \langle |w|^q \rangle
\]

Using \( ab \leq \frac{\gamma}{4} a^2 + \frac{1}{\gamma} b^2 \) (\( \eta > 0 \)), we obtain

\[
|F_2| \leq (q - 2) \int_s^\tau \langle |w|^{q-2} |\nabla |w||b_m||w| \rangle dt = (q - 2) \int_s^\tau \langle |w|^{\frac{q-2}{2}} |\nabla |w||b_m||w|^{\frac{q}{2}} \rangle dt \leq
(q - 2) \left[ \eta \int_s^\tau \langle |w|^{q-2} |\nabla |w|^2 \rangle dt + \frac{1}{4\eta} \int_s^\tau \left< \left( |b_m||w|^{\frac{q}{2}} \right)^2 \right> dt \right] \leq
(q - 2) \left[ \eta J_q + \frac{\beta + 1/m}{4\eta} \frac{q^2}{4} J_q + \frac{1}{4\eta} \int_s^\tau g(t) \langle |w|^q \rangle dt \right].
\]
Finally, integrating by parts, we see that $S_3 = \int_s^T h'(t)\langle |w|^q \rangle dt$. Thus, identity $S = S_1 + S_2 + S_3$ transforms into

\[
\frac{1}{q} \langle |w(\tau,s,\cdot)|^q \rangle - \frac{1}{q} \langle |\nabla f|^q \rangle - \int_s^\tau h'(t)\langle |w|^q \rangle dt + I_q + (q - 2)J_q = F_1 + F_2,
\]

and, in view of the above estimates on $|F_1|, |F_2|$ and elementary inequality $I_q \geq J_q$, implies

\[
\frac{1}{q} \langle |w(\tau,s,\cdot)|^q \rangle + M J_q \leq \frac{1}{q} \langle |\nabla f|^q \rangle + \int_s^\tau h'(t)\langle |w|^q \rangle dt + \left( \frac{q - 2}{4q} + \frac{1}{\gamma} \right) \int_s^\tau g(t)\langle |w|^q \rangle dt,
\]

where

\[
M := 1 - \frac{d\gamma}{4} + q - 2 - (q - 2) \left( \eta + \frac{\beta + 1/m}{16\eta} q^2 \right) - \frac{\beta + 1/m}{4} q^2.
\]

The maximum of $M = M(\gamma, \eta)$, attained at $\gamma_* = q\sqrt{\frac{\beta + 1/m}{d}}$, $\eta_* = q\sqrt{\frac{\beta + 1/m}{4d}}/4$, is positive starting from some $m$ if and only if $\beta < \frac{1}{q^2}\omega^2_\gamma$. We select

\[
\gamma = \gamma_*, \quad \eta = \eta_*, \quad h'(t) > -\left( (q - 2)/4\eta_* + 1/\gamma_* \right) g(t).
\]

Then (16) immediately yields (15) ($\Rightarrow$ Lemma 3(2)). To obtain estimate (14), we apply the Sobolev embedding theorem to $\frac{1}{q} J_q = \int_s^\tau \langle |\nabla |^2 \rangle dt$ to get $\frac{1}{q} J_q \geq C_0 \int_s^\tau \langle |w|^{2d/(d - 2)} \rangle dt^{d-2/d}$. Substituting this in (16) and integrating in $s$, we obtain (14) ($\Rightarrow$ Lemma 3(1)).

3.2. **Proof of Lemma 4**. Set $r = r_{m,n} := v_{m} - v_n$, where $v_{m}$ ($v_n$) is the solution of Cauchy problem (10). Then $r$ satisfies

\[
\partial_t r = \Delta r + b_m(t, x) \cdot \nabla r + (b_{m}(t, x) - b_{n}(t, x)) \cdot \nabla v_n + h'(t)r.
\]

Denote $\varphi := r |r|^{p-2}$. We multiply equation (17) by $\varphi$ and integrate over $D_T \times \mathbb{R}^d$ (recall $D_T := \{(s, t) \in \mathbb{R}^2 : 0 \leq s \leq t \leq T\}$) to get identity

\[
\frac{1}{p} \int_0^T \langle \eta(T, s, \cdot) \rangle^2 ds + \frac{4(p - 1)}{p^2} \int_{D_T} \|\nabla \eta\|^2_{L^2} dt ds = \frac{2}{p} \int_{D_T} \langle \nabla \eta, b_m \eta \rangle dt ds + \int_{D_T} \langle \eta \rangle^{1 - \frac{2}{p}} \langle b_m - b_n \rangle \cdot \nabla v_n \rangle dt ds + \int_{D_T} h'(t)\langle \eta^2 \rangle dt ds
\]

(note that by definition $\eta(s, s, \cdot) \equiv 0$). We estimate the right-hand side of (18) as follows:

1) Using inequality $ab \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon}$ ($\varepsilon > 0$) and condition (C2), we obtain:

\[
\int_{D_T} \langle \nabla \eta, b_m \eta \rangle dt ds \leq \varepsilon \int_{D_T} \langle (b_m \eta)^2 \rangle dt ds + \frac{1}{4\varepsilon} \int_{D_T} \|\nabla \eta\|^2_{L^2} dt ds
\]

\[
\leq \varepsilon (\beta + \frac{1}{m}) \int_{D_T} \langle |\nabla \eta|^2 \rangle dt ds + \varepsilon \int_{D_T} g(t)\langle \eta^2 \rangle dt ds + \frac{1}{4\varepsilon} \int_{D_T} \|\nabla \eta\|^2_{L^2} dt ds.
\]
2) Using inequalities $|b_m - b_n| \leq |b_m| + |b_n|$, $ab \leq \delta a^2 + \frac{1}{\delta} b^2$ ($\delta > 0$), and condition (C2), we find

$$\left| \int_{D_T} \langle |\eta| \rangle^{\frac{p-1}{p}} \delta |\eta| \cdot \nabla v \rangle dt ds \right| \leq \int_{D_T} \langle |b_m - b_n| |\eta| \rangle^{\frac{p-1}{p}} |\nabla v| dt ds$$

$$\leq \delta \int_{D_T} \langle (b_m \eta)^2 \rangle dt ds + \delta \int_{D_T} \langle (b_n \eta)^2 \rangle dt ds + \frac{1}{4\delta} \int_{D_T} \langle |\eta|^2 \rangle^2 dt ds$$

$$\leq 2\delta \left( \beta + \frac{1}{\min\{m, n\}} \right) \int_{D_T} \langle |\nabla \eta|^2 \rangle dt ds + \int_{D_T} \langle g(t) |\eta|^2 \rangle dt ds + \frac{1}{2\delta} \int_{D_T} \langle |\eta|^2 \rangle^2 dt ds.$$  

We choose a sufficiently large $m_0$, so that $\beta' := \beta + \frac{1}{m_0} < 4$. In what follows, we take $n, m > m_0$ and, in order to simplify the notation, re-denote $\beta'$ by $\beta$. We obtain from identity (18) and estimates 1), 2)

$$\frac{1}{p} \int_0^T \|\eta(T, s, \cdot)\|_2^2 ds + \left( \frac{4(p-1)}{p^2} - \frac{2}{p} \left( \varepsilon \beta + \frac{1}{4\varepsilon} \right) - 2\beta \delta \right) \int_{D_T} \langle |\nabla \eta|^2 \rangle dt ds$$

$$- \left( \frac{2}{p} \varepsilon + 2\delta \right) \int_{D_T} g(t) |\eta|^2 dt ds$$

$$\leq \int_{D_T} \langle h'(t) |\eta|^2 \rangle dt ds \leq \frac{1}{2\delta} \int_{D_T} \langle |\eta|^2 \rangle^2 dt ds.$$  

Choose

$$\varepsilon := \frac{1}{2\sqrt{\beta}}, \quad \delta := \frac{1}{2\beta} \left( \frac{4(p-1)}{p^2} - \frac{2}{p} \sqrt{\beta} \right).$$

Then the coefficient of $\int_{D_T} \langle |\nabla \eta|^2 \rangle dt ds$ in (19) can be computed as follows:

$$\frac{4(p-1)}{p^2} - \frac{2}{p} \left( \varepsilon \beta + \frac{1}{4\varepsilon} \right) - 2\beta \delta = \frac{4(p-1)}{p^2} - \frac{2}{p} \sqrt{\beta} - 2\beta \delta = \frac{1}{p^2}.$$  

Next, since $p_0 > 1$, we can choose $k$ so that

$$\frac{4(p_0-1)}{p_0^2} - \frac{2}{p_0} \sqrt{\beta} \geq \frac{2}{p_0^2}.$$  

(21)

It is immediate that the latter inequality also holds if we replace $p_0$ and $m_0$ with any $p > p_0$ and $m > m_0$. Then, by (21) and our choice of $\delta$,

$$\delta \geq \frac{1}{2\beta p^k}.$$  

(22)

Further, it is also seen that with the above choice of $\varepsilon$ and $\delta$ there exist $c_1, c_2 > 0$ independent of $p \geq p_0, m \geq m_0$ such that for $h(t) := -c_1 \int_0^t g(\theta) d\theta$ we have

$$- \left( \frac{2}{p} \varepsilon + 2\delta \right) \int_{D_T} g(t) |\eta|^2 dt ds - \int_{D_T} h'(t) |\eta|^2 dt ds \geq c_2 \int_{D_T} g(t) |\eta|^2 dt ds ($$\geq 0$).

Now, we obtain from (19), using (20), (22) and the last inequality, that

$$\frac{1}{p^2} \int_{D_T} \langle |\nabla \eta|^2 \rangle dt ds \leq \beta p^k \int_{D_T} \langle |\eta|^2 \rangle^2 dt ds.$$  

(23)
Using Hölder inequality with \( \sigma, \sigma' > 1, \frac{1}{\sigma} + \frac{1}{\sigma'} = 1 \), we obtain (recall that \( |\eta| = |r|^\frac{2}{p'} \)):

\[
\int_{D_T} \langle |\eta|^{2-\frac{4}{p}} |\nabla v_n|^2 \rangle dt ds \leq \left( \int_{D_T} \langle |\nabla v_n|^{2\sigma} \rangle dt ds \right)^{\frac{1}{\sigma}} \left( \int_{D_T} \langle |\eta|^{2-\frac{4}{p}} \rangle \langle |\eta|^{2\sigma'} \rangle \right)^{\frac{1}{\sigma'}}
\]

\[
= \|\nabla v_n\|^2_{L^{2\sigma}(D_T \times \mathbb{R}^d)} \|\eta\|^{2(1-\frac{4}{p})}_{L^{2(1-\frac{4}{p})\sigma'}(D_T \times \mathbb{R}^d)} = \|\nabla v_n\|^2_{L^{2\sigma}(D_T \times \mathbb{R}^d)} \|r\|^{p-2}_{L^{p(\sigma - 2)\sigma'}(D_T \times \mathbb{R}^d)}.
\]

Further, applying Sobolev inequality in the spacial variables, we get

\[
\int_{D_T} \langle |\nabla \eta|^2 \rangle dt ds \geq c_0 \left( \int_{D_T} \langle |\eta|^{\frac{d}{2-\frac{4}{p}}} \rangle \|\nabla v_n\|^2 \right)^{\frac{d-2}{d}} = c_0 \left( \int_{D_T} \langle |r|^{\frac{d}{2-\frac{4}{p}}} \rangle \|\nabla v_n\|^2 \right)^{\frac{d-2}{d}} = c_0 \|r\|^p_{L^{\frac{dp}{d-2}}(D_T \times \mathbb{R}^d)}
\]

where \( c_0 > 0 \) is the best constant in Sobolev inequality. By the last two estimates, (23) transforms into

\[
\frac{1}{p^k c_0 \|r\|^p_{L^{\frac{dp}{d-2}}(D_T \times \mathbb{R}^d)}} \leq \frac{1}{p^k} \int_{D_T} \langle |\nabla \eta|^2 \rangle dt ds \leq \beta p^k \int_{D_T} \langle |\eta|^{2-\frac{4}{p}} |\nabla v_n|^2 \rangle dt ds \leq \beta p^k \|\nabla v_n\|^2_{L^{2\sigma}(D_T \times \mathbb{R}^d)} \|r\|^{p-2}_{L^{p(\sigma - 2)\sigma'}(D_T \times \mathbb{R}^d)}
\]

i.e.

\[
\|r\|^p_{L^{\frac{dp}{d-2}}(D_T \times \mathbb{R}^d)} \leq C_0 \beta p^{2k} \|\nabla v_n\|^2_{L^{2\sigma}(J \times \mathbb{R}^d)} \|r\|^{p-2}_{L^{p(\sigma - 2)\sigma'}(D_T \times \mathbb{R}^d)},
\]

where \( C_0 = c_0^{-1} < \infty \). Finally, recalling that \( r := v_n - v_m \), and \( \beta \) is the re-denoted \( \beta + \frac{1}{m_0} \), we arrive at the required estimate (3) in Lemma 3.

### 3.3. Proof of Corollary 5

The proof of Corollary 5 follows closely the proof of [KS, Lemma 7]. Fix an arbitrary \( q > \max \{ d - 2, 2 \} \) and select \( \sigma := \frac{d}{2-\frac{4}{p}} > \frac{d}{2} \). Define \( 1 < \sigma' < \frac{d}{2} \) via \( \frac{1}{\sigma} + \frac{1}{\sigma'} = 1 \). We are going to iterate the inequality of Lemma 3.

\[
\|v_m - v_n\|_{L^{\frac{dp}{d-2}}(D_T \times \mathbb{R}^d)} \leq \left( C_0 \left( \beta + \frac{1}{m_0} \right) \|\nabla v_m\|^2_{L^{2\sigma}(D_T \times \mathbb{R}^d)} \right)^{\frac{1}{p}} (p^{2k})^{\frac{1}{p}} \|v_m - v_n\|_{L^{p(\sigma - 2)\sigma'}(D_T \times \mathbb{R}^d)}\]

(24)

where \( v_m, v_n \) solves (10).

First, note that by our assumption on \( \beta \) we can apply Lemma 3(1) to obtain \( \|\nabla v_m\|_{L^{2\sigma}(D_T \times \mathbb{R}^d)} \leq \tilde{C} \|\nabla f\|_q \) for some \( \tilde{C} < \infty \) independent of \( m \) and \( f \). Then (24) yields

\[
\|v_m - v_n\|_{L^{\frac{dp}{d-2}}(D_T \times \mathbb{R}^d)} \leq D \left( p^{2k} \right)^{\frac{1}{p}} \|v_m - v_n\|_{L^{p(\sigma - 2)\sigma'}(D_T \times \mathbb{R}^d)},
\]

(25)

where \( D < \infty \). Choose any \( p_0 > \frac{2}{2-\frac{4}{p}} \), and construct a sequence \( \{p_l\}_{l \geq 0} \) by successively assuming \( \sigma'(p_l - 2) = p_0, \sigma'(p_{l+1} - 1) = \frac{p_0 d}{d-2}, \sigma'(p_{l+2} - 1) = \frac{p_0 d}{d-2} \) etc, so that

\[
p_l = (a - 1)^{-1}(a' \left( \frac{p_0}{\sigma'} + 2 \right) - a^{-1} \frac{p_0}{\sigma'} - 2), \quad a := \frac{1}{\sigma'} \frac{d}{d - 2} > 1.
\]

(26)

Clearly,

\[
c_1 a' \leq p_l \leq c_2 a', \quad \text{where} \quad c_1 := p_1 a^{-1}, \quad c_2 := c_1 (a - 1)^{-1},
\]

(27)
so \( p_l \to \infty \) as \( l \to \infty \). We now iterate inequality (25), starting with \( p = p_0 \), to obtain
\[
\|v_m - v_n\|_{L}^{p_i}(D_T \times \mathbb{R}^d) \leq D^\alpha \Gamma_l \|v_m - v_n\|_{L}^{q}(D_T \times \mathbb{R}^d),
\]
where \( \gamma_l := \left(1 - \frac{2}{p_l}ight) \ldots \left(1 - \frac{2}{p_1}\right) \),
\[
\alpha_l := \frac{1}{p_1} \left(1 - \frac{2}{p_2}\right) \left(1 - \frac{2}{p_3}\right) \ldots \left(1 - \frac{2}{p_l}\right) + \ldots + \frac{1}{p_{l-1}} \left(1 - \frac{2}{p_l}\right) + \frac{1}{p_l},
\]
\[
\Gamma_l := \left(p_l^{-1} p_{l-1}^{-1} \ldots p_1^{-1}\right)^{2k}.
\]
We wish to take \( l \to \infty \) in (28); since \( p_l \to \infty \) as \( l \to \infty \), this would yield the required inequality (9) provided that sequences \( \{\alpha_l\} \), \( \{\Gamma_l\} \) are bounded from above, and \( \{\gamma_l\} \) is bounded from below by a positive constant. Indeed, we can compute \( \alpha_l = a_l - \frac{1}{p_l(a_l-1)} \), \( \gamma_l = p_0 \frac{a_l^{-1}}{a_l-1} \), and note that, in view of (26),
\[
\sup_l \alpha_l \leq \left(\frac{p_0}{\sigma'} + 2 - \frac{p_0(d-2)}{d}\right)^{-1} < \infty, \quad \sup_l \gamma_l < \infty,
\]
\[
\inf \gamma_l > \left(1 - \frac{\sigma'(d-2)}{d}\right) \left(1 - \frac{\sigma'(d-2)}{d} + \frac{2\sigma'}{p_0}\right)^{-1} > 0.
\]
Further, it is not difficult to see (cf. (26)) that \( \Gamma_{l}^{1/2k} = p_l^{-1} p_{l-1}^{-1} \ldots p_1^{-1} \). Then by (27)
\[
\Gamma_{l}^{1/2k} \leq (c_1 a_l) (c_2 a_l)^{-1} (c_1 a_{l-1}) (c_2 a_{l-1})^{-1} \ldots (c_1 a_1) (c_2 a_1)^{-1} = \left(c_1^{(a_1-1)(a_1'-1)} \ldots c_2^{(a_1-1)(a_1'-1)} \right)^{-1} < \infty.
\]
It follows from estimates (29), (30) and (31) that we can take \( l \to \infty \) in (28), which then yields (9) in Corollary 5.

3.4. **Proof of Lemma 6** Let \( v_m \) be the solution of Cauchy problem (10). Clearly, \( \{v_m\} \subset L^q(D_T \times \mathbb{R}^d), q \geq 2 \). We need to prove that \( \{v_m\} \) is fundamental \( L^q(D_T \times \mathbb{R}^d), q \geq 2 \), for some \( h \) independent of \( m \).

**Claim 7.** Suppose \( \{v_m\} \) is fundamental in \( L^2(D_T \times \mathbb{R}^d) \). Then \( \{v_m\} \) is fundamental in \( L^q(D_T \times \mathbb{R}^d), q > 2 \).

**Proof of Claim 7.** By Hölder inequality
\[
\|v_m - v_n\|_{L^q(D_T \times \mathbb{R}^d)} : = \int_{(s,t) \in D_T} (|v_m - v_n|^q) dsdt = \int_{(s,t) \in D_T} (|v_m - v_n|^{q-1} |v_m - v_n|) dsdt \\
\leq \left( \int_{(s,t) \in D_T} (|v_m - v_n|^{2(q-1)}) dsdt \right)^{1/2} \left( \int_{(s,t) \in D_T} (|v_m - v_n|^2) dsdt \right)^{1/2}
\]
The first multiple in the right-most part is bounded uniformly in \( m, n \) by estimate (13) (there we need \( 2(q - 1) > (1 - \sqrt{\beta/4})^{-1} \), which is true by our assumption \( \beta < 1 \)). The second multiple converges to 0 as \( m, n \to \infty \) because \( \{v_m\} \) is fundamental in \( L^2(D_T \times \mathbb{R}^d) \). So, \( \|v_m - v_n\|_{L^2(D_T \times \mathbb{R}^d)} \to 0 \) as \( m, n \to \infty \).

Therefore, it suffices to show that \( \{v_m\} \) is fundamental in \( L^2(D_T \times \mathbb{R}^d) \). We do it in three steps.

**Step 1.** Fix \( k \geq 1 \), and define

\[
\rho_\delta(x) := (1 + \delta|x|^2)^{-k}, \quad \delta > 0, \quad x \in \mathbb{R}^d,
\]

so that \( \rho_\delta(x) \to 0 \) as \( x \to \infty \), and for every \( \Omega \subseteq \mathbb{R}^d \), \( \rho_\delta \to 1 \) uniformly on \( \Omega \) as \( \delta \to 0 \).

Let us show that for any \( \varepsilon > 0 \) there is a sufficiently small \( \delta > 0 \) such that for all \( m \)

\[
\langle v_m^2(\tau, s, \cdot)(1 - \rho_\delta) \rangle < \varepsilon, \quad (s, \tau) \in D_T.
\]  

(Informally, \( v_m \) ‘do not run away to infinity’ as \( m \to \infty \).)

It is easy to verify that

\[
|\nabla \rho_\delta(x)| \leq 2k\sqrt{\delta}\rho_\delta(x), \quad \left| \frac{(\nabla \rho_\delta(x))^2}{1 - \rho_\delta(x)} \right| \leq 4k^2\delta\rho_\delta(x) \quad \text{for all} \ x \in \mathbb{R}^d.
\]  

The estimate (32) follows from the next

**Claim 8.** We can choose \( h \) in (10) in such a way that, for all \( m \),

(a)

\[
\langle v_m^2(\tau, s, \cdot) \rangle + C_0 \int_s^T \langle (\nabla v_m)^2 \rangle dt \leq \langle f^2 \rangle, \quad (s, \tau) \in D_T
\]

where \( 0 < C_0 < \infty \) is independent of \( m \),

(b)

\[
\langle v_m^2(\tau, s, \cdot)(1 - \rho_\delta) \rangle + C_0 \int_s^T \langle (\nabla v_m)^2(1 - \rho_\delta) \rangle dt \leq \langle f^2(1 - \rho_\delta) \rangle
\]

\[
+ C_1k\sqrt{\delta}\left( \int_s^T \langle v_m^2 \rho_\delta \rangle dt + \int_s^T \langle (\nabla v_m)^2 \rho_\delta \rangle dt \right), \quad (s, \tau) \in D_T,
\]

where \( C_1 < \infty \) is independent of \( m \).

**Proof of (32).** By the estimate of Claim 8(b), applying in its last term estimates \( |\rho_\delta(x)| \leq 1, \int_s^\tau \langle v_m^2 \rangle dt \leq (\tau - s)C\langle f^2 \rangle \) (cf. (13)) and Claim 8(a), we obtain

\[
\langle v_m^2(\tau, s, \cdot)(1 - \rho_\delta) \rangle \leq \langle f^2(1 - \rho_\delta) \rangle + C_1k\sqrt{\delta}(C(\tau - s) + C_0^{-1})\langle f^2 \rangle, \quad (s, \tau) \in D_T.
\]

Since \( \rho_\delta \to 1 \) uniformly on the support of \( f \in C_0^\infty(\mathbb{R}^d) \) as \( \delta \to 0 \), the right-hand side of the last estimate can can be made as small as needed by taking \( \delta \) to be sufficiently small, i.e. we have (32). \( \square \)
**Proof of Claim 8.** We will prove (a) in the course of proof of (b). Denote
\[ A := \int_s^t \langle (\nabla v_m)^2 (1 - \rho_\delta) \rangle dt \geq 0, \quad B := \int_s^t \left\langle \frac{v_m^2 (\nabla \rho_\delta)^2}{1 - \rho_\delta} \right\rangle dt \geq 0, \]
\[ G := \int_s^t \langle g(t)v_m^2 (1 - \rho_\delta) \rangle dt, \quad R := \int_s^t \langle \nabla v_m^2 \nabla \rho_\delta \rangle dt. \]

To prove (b), multiply the equation in [10] by \( v_m (1 - \rho_\delta) \) and integrate to get
\[ M := \int_s^t \frac{d}{dt} \langle v_m^2 (1 - \rho_\delta) \rangle dt =
2 \int_s^t \langle v_m \Delta v_m (1 - \rho_\delta) \rangle dt + 2 \int_s^t \langle v_m b_m \nabla v_m (1 - \rho_\delta) \rangle dt + 2 \int_s^t \langle v_m^2 h'(t)(1 - \rho_\delta) \rangle dt
=: M_1 + M_2 + M_3. \quad (34) \]

We have
1) \( M = \langle v_m^2 (\tau, s, \cdot) (1 - \rho_\delta) \rangle - \langle f^2 (1 - \rho_\delta) \rangle, \)
2) \( M_1 = -2A - R, \)
3) \( M_2 \leq \left( \gamma + \beta + \frac{1}{\gamma} \right) A + \frac{\beta + \frac{1}{\gamma^2}}{2} \left( \frac{1}{4} + \frac{1}{2\pi} \right) B + G. \)

Identities 1), 2) are self-evident. Estimate 3) is obtained as follows. Using elementary inequality \( 2ab \leq \gamma a^2 + \frac{1}{\gamma} b^2 \) \((\gamma > 0)\) and condition \((C_2)\), we find
\[ M_2 = 2 \int_s^t \langle v_m b_m \nabla v_m (1 - \rho_\delta) \rangle dt \leq \gamma A + \frac{1}{\gamma} \int_s^t \langle b_m^2 (v_m \sqrt{1 - \rho_\delta})^2 \rangle dt \]
\[ \leq \gamma A + \frac{\beta + 1}{\gamma} \int_s^t \langle (\nabla (v_m \sqrt{1 - \rho_\delta}))^2 \rangle dt + \frac{1}{\gamma} G \]

We obtain 3) after noting that
\[ \int_s^t \langle (\nabla (v_m \sqrt{1 - \rho_\delta}))^2 \rangle dt = \int_s^t \left\langle (\nabla v_m)^2 (1 - \rho_\delta) - 2 \nabla v_m \cdot \frac{\nabla \rho_\delta}{2 \sqrt{1 - \rho_\delta}} + \frac{1}{4} v_m^2 \frac{(\nabla \rho_\delta)^2}{1 - \rho_\delta} \right\rangle dt \]
\[ \leq (1 + \epsilon) A + \left( \frac{1}{4} + \frac{1}{4\epsilon} \right) B, \]
where we used inequality \( 2ab \leq \epsilon a^2 + \frac{1}{\epsilon} b^2 \) \((\epsilon > 0)\).

We now substitute 1), 2) and 3) into equality \( M = M_1 + M_2 + M_3 \) to get
\[ \langle v_m^2 (\tau, s, \cdot) (1 - \rho_\delta) \rangle + \left( 2 - \gamma + \frac{\beta - \frac{1}{\gamma}}{\gamma} (1 + \epsilon) \right) A \]
\[ \leq \langle f^2 (1 - \rho_\delta) \rangle + |R| + \frac{\beta + \frac{1}{\gamma^2}}{2} \left( \frac{1}{4} + \frac{1}{2\pi} \right) B + \frac{1}{\gamma} G + M_3. \quad (35) \]

Since \( \beta < 1 \), there exists \( \gamma > 0 \) such that starting from some \( m \), for all \( \epsilon > 0 \) sufficiently small \( 2 - \gamma - \frac{\beta + \frac{1}{\gamma}}{\gamma} (1 + \epsilon) > 0 \). Let us fix such \( \gamma \), and select \( h(t) := -\frac{2}{\gamma} \int_0^t g(\theta) d\theta \), so that \( \frac{1}{\gamma} G + M_3 = 0. \)

**Remark 4.** Up to this moment, we can repeat everything replacing \( \rho_\delta \) with 0 (then \( R = 0, B = 0 \)), in which case estimate \((35)\) would imply Claim \((8)\)(a).
It remains to estimate $|R|$ and $B$. We have

$$|R| = 2 \int_s^\tau \langle v_m \nabla v_m, \nabla \rho_\delta \rangle dt$$

$$\leq \int_s^\tau \langle (v_m)^2 |\nabla \rho_\delta| \rangle dt + \int_s^\tau \langle (\nabla v_m)^2 |\nabla \rho_\delta| \rangle dt \quad \text{(by the first estimate in (33))}$$

$$\leq 2k\sqrt{3} \left( \int_s^\tau \langle v_m^2 \rho_\delta \rangle dt + \int_s^\tau \langle (\nabla v_m)^2 \rho_\delta \rangle dt \right).$$

Using the second estimate in (33), we obtain $B \leq 2k^2 \delta \int_s^\tau \langle v_m^2 \rho_\delta \rangle dt$. Assertion (b) now follows from (35) and the estimates on $|R|$ and $B$. \qed

**Step 2.** Denote by $1_\alpha$ the characteristic function of open ball $B_0(\alpha) \subset \mathbb{R}^d$ of radius $\alpha$ centered at 0. Set $r_{m,n} := v_m - v_n$. Let us show that

$$\langle 1_\alpha r_{m,n}^2(\tau, s, \cdot) \rangle < \varepsilon \quad \text{for all } m, n \text{ sufficiently large.} \quad (36)$$

In what follows, we omit indices $m, n$ wherever possible: $r := r_{m,n}$. Define for $\sigma > 0$ and $y \geq 0$

$$\xi(y) = \begin{cases} 1 & y \leq \alpha \\ 1 - \sigma(y - \alpha) & \alpha < y \leq \alpha + \frac{1}{\sigma} \\ 0 & y > \alpha + \frac{1}{\sigma} \end{cases}$$

Set $\eta(x) := \xi^2(|x|)$ ($x \in \mathbb{R}^d$). Clearly,

$$|\nabla \eta| \leq 2\sigma, \quad \frac{(\nabla \eta)^2}{\eta} \leq 4\sigma^2 \quad \text{on } B_0(\alpha + 1/\sigma). \quad (37)$$

It is also clear that

$$\langle \eta r(\tau, s, \cdot)^2 \rangle < \varepsilon \quad \implies \quad \langle 1_\alpha r^2(\tau, s, \cdot) \rangle < \varepsilon.$$

Thus, to prove (36), it suffices to show that the inequality in the left-hand side holds true. Indeed, $r = v_m - v_n$ satisfies

$$\partial_t r = \Delta r + b_m \cdot \nabla r + (b_m - b_n) \cdot \nabla v_n + h'(t)r, \quad r(s, s, \cdot) \equiv 0,$$

so, multiplying both sides of the first identity by $\eta r$ and integrating over $[s, \tau] \times \mathbb{R}^d$, we obtain

$$\int_s^\tau \left\langle \eta \frac{\partial r}{\partial t} \right\rangle dt = \int_s^\tau \langle \eta \Delta r \rangle dt + \int_s^\tau \langle \eta r b_m, \nabla r \rangle dt + \int_s^\tau \langle \eta r (b_m - b_n), \nabla v_n \rangle dt + \int_s^\tau h'(t)\langle \eta r^2 \rangle dt. \quad (38)$$

In the right-hand side of (38): we integrate by parts in the first term, apply inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$ to the second term and apply Hölder inequality to the third term to obtain

$$\frac{1}{2} \langle \eta r^2(\tau, s, \cdot) \rangle \leq - \frac{1}{2} \int_s^\tau \langle \eta (\nabla r)^2 \rangle - \int_s^\tau \langle r \nabla \eta \nabla r \rangle + \frac{1}{2} \int_s^\tau \langle \eta b_m^2 r^2 \rangle dt + \int_s^\tau \langle \eta r^2 (b_m - b_n) \rangle dt + \frac{1}{2} \int_s^\tau \langle \eta (\nabla v_n)^2 \rangle dt + \frac{1}{2} \int_s^\tau h'(t)\langle \eta r^2 \rangle dt \quad (39)$$
or

\[ K \leq -\frac{1}{2}K_1 - K_2 + \frac{1}{2}K_3 + K_4 + K_5. \]

Our goal is to show that \((K_{m,n} = K) = \frac{1}{2}\langle \eta \tau^2(\tau, s, \cdot) \rangle \rightarrow \infty \) as \( m, n \rightarrow \infty \).

The following estimate follows directly from the first estimate in (37) and inequality 2ab \( \leq a^2 + b^2 \):

\[ |K_2| \leq \sigma \int_{s}^{T} (\langle r^2 \rangle + \langle (\nabla r)^2 \rangle) dt. \]

Note that \( \langle r^2(\tau, s, \cdot) \rangle \leq 4C\langle f^2 \rangle \) and \( \int_{s}^{T} (\langle r^2 \rangle) \leq 4C_1\langle f^2 \rangle \) by Claim 8(a), where constants \( C, C_1 < \infty \) are independent of \( m, n \) (recall that \( r := v_m - v_n \)). Therefore,

\[ |K_2| \leq \hat{C}\sigma, \quad (40) \]

where \( \hat{C} = 4C\langle f^2 \rangle(\tau - s) + 4C_1\langle f^2 \rangle \).

Next, using (C2) and the second estimate in (33), we obtain

\[ K_3 \leq (\beta + 1/m) \int_{s}^{T} \langle (\nabla (\sqrt{\eta}r))^2 \rangle dt + \int_{s}^{T} g(t)\langle \eta \tau^2 \rangle dt \leq \]

\[ (\beta + 1/m) \int_{s}^{T} (\sigma^2(\tau^2) + \langle (\nabla \eta \nabla r) + \langle \eta (\nabla r)^2 \rangle \rangle dt + \int_{s}^{T} g(t)\langle \eta \tau^2 \rangle dt \leq \]

\[ (\beta + 1/m) \int_{s}^{T} \sigma^2(\tau^2) dt + (\beta + 1/m)K_2 + (\beta + 1/m)K_1 + \int_{s}^{T} g(t)\langle \eta \tau^2 \rangle dt. \quad (41) \]

Let us show that \( K_4 = K_{4,m,n} \rightarrow 0 \) as \( m, n \rightarrow \infty \). Since \( |\eta(x)| \leq 1 \), \( \int_{s}^{T} \langle (\nabla \eta)^2 \rangle dt \leq C_1\langle f^2 \rangle \) by Claim 8(a), and \( \|r(\tau, s, \cdot)\|_{\infty} \leq 2C\|f\|_{\infty} \) by (12), for some \( C, c > 0 \) independent of \( n, m \), we have

\[ K_4 \leq 2c\|f\|_{\infty} \left( \int_{s}^{T} \langle \eta (b_m - b_n)^2 \rangle dt \right)^{\frac{1}{2}} C_1^{1/2}/(f^2)^{1/2}; \]

the right-hand side tends to 0 as \( m, n \rightarrow \infty \) since, by (C1) \( b_m \rightarrow b \) in \( L_{loc}^2([0, \infty) \times \mathbb{R}^d) \), \( \eta \) has compact support in \([0, \infty) \times \mathbb{R}^d \), so \( \eta(b_m - b_n) \rightarrow 0 \) in \( L^2([0, T] \times \mathbb{R}^d) \). Therefore, \( K_4 \rightarrow 0 \) as \( m, n \rightarrow \infty \).

We now combine the above estimates on \( K, K_i \)'s. In view of (41) \( K \leq -\frac{1}{2}K_1 - K_2 + \frac{1}{2}K_3 + K_4 + K_5 \) implies

\[ K \leq \left( \frac{\beta + 1/m}{2} - \frac{1}{2} \right) K_1 + \left( 1 + \frac{\beta + 1/m}{2} \right) |K_2| \]

\[ + (\beta + 1/m) \int_{s}^{T} \sigma^2(\tau^2) dt + \int_{s}^{T} g(t)\langle \eta \tau^2 \rangle dt + K_4 + K_5. \]

We select \( h'(t) = -g(t) \), so \( \int_{s}^{T} g(t)\langle \eta \tau^2 \rangle dt + K_5 = 0 \). Therefore, we can re-write the last estimate as

\[ K + \left( \frac{1}{2} - \frac{\beta + 1/m}{2} \right) K_1 \leq \left( 1 + \frac{\beta + 1/m}{2} \right) |K_2| + (\beta + 1/m) \int_{s}^{T} \sigma^2(\tau^2) dt + K_4, \]
where, since $\beta < 1$, the coefficient of $K_1$ is positive provided that $m$ is sufficiently large. Now, in view of (\ref{claim}), Claim \ref{claim}, and since $\beta < 1$ we have, starting from some $m$,
\[
K + \left(\frac{1}{2} - \frac{\beta + 1/m}{2}\right) K_1 \leq \frac{3}{2} \hat{C}\sigma + 4\sigma^2 C(f^2)(\tau - s) + K_4.
\] (42)
We can choose a sufficiently small $\sigma$ in the definition of function $\eta$ and then use the fact that $K_4 = K_{4,m,n} \to 0$ as $m, n \to \infty$ to make the right-hand side of (42) as small as we wish, for all $m, n$ sufficiently large. Thus, $(K_{m,n} = ) K = \frac{1}{2}(\eta^2(\tau, s, \cdot)) \to 0$ as $m, n \to \infty$; this yields (35).

**Step 3.** We now combine the results of Step 1 and Step 2. First, let us show that for any $\varepsilon > 0$ there exist $\alpha > 0$ such that
\[
\langle (1 - 1_\alpha)r_{m,n}(\tau, s, \cdot)^2 \rangle < \varepsilon \quad \text{for all } m, n.
\] (43)
Indeed, by the result of Step 1 we can find $\delta > 0$ such that
\[
\langle (1 - \rho_\delta) r_{m,n}(\tau, s, \cdot)^2 \rangle \leq 2\langle v_m(\tau, s, \cdot)^2(1 - \rho_\delta) \rangle + 2\langle v_m(\tau, s, \cdot)^2(1 - \rho_\delta) \rangle < \frac{\varepsilon}{3},
\] (44)
Choose $\alpha = \alpha(\delta)$ such that $|1 - \rho_\delta(x)| > \frac{1}{3}$ for all $x \in \mathbb{R}^d \setminus B_0(\alpha)$, and so $(1 - 1_\alpha(x)) < 3(1 - \rho_\delta(x)), x \in \mathbb{R}^d$. Then
\[
\langle (1 - 1_\alpha)r_{m,n}(\tau, s, \cdot)^2 \rangle < 3\langle (1 - \rho_\delta(x)) r_{m,n}(\tau, s, \cdot)^2 \rangle < 3 \cdot \frac{\varepsilon}{3} = \varepsilon,
\]
as needed.

Combining (35) with the result of Step 2, we obtain that $\langle r_{m,n}(\tau, s, \cdot)^2 \rangle < 2\varepsilon$ for all $m, n$ sufficiently large. Since $\varepsilon > 0$ was chosen arbitrarily, sequence $\{v_m(\tau, s, \cdot)\}$ is fundamental in $L^2(\mathbb{R}^d)$. Therefore, we can set
\[
v_0(\tau, s, \cdot) := L^2(\mathbb{R}^d) - \lim_{m \to \infty} v_m(\tau, s, \cdot), \quad (s, \tau) \in D_T,
\]
where, recall, $D_T = \{(s, t) \in \mathbb{R}^2 : 0 \leq s \leq t \leq T\}$ for fixed $T > 0$. The limit $v_0$ belongs to $L^\infty(D_T, L^2(\mathbb{R}^d))$ by (13) (we need to have $(1 - \sqrt{\beta/4})^{-1} < 2$ there, which is true because $\beta < 1$). In particular, since $D_T$ is compact, $v_0 \in L^2(D_T \times \mathbb{R}^d)$. We have proved that for each $(s, \tau) \in D_T$,
\[
\langle (v_m(\tau, s, \cdot) - v_0(\tau, s, \cdot))^2 \rangle \to 0 \quad \text{as } m \to \infty.
\]
Applying the Dominated Convergence Theorem to sequence $D_T \ni (s, \tau) \mapsto \langle (v_m(\tau, s, \cdot) - v_0(\tau, s, \cdot))^2 \rangle$, cf. (11) and (13), we obtain that $v_m \to v_0$ in $L^2(D_T \times \mathbb{R}^d)$, as needed.

3.5. **Proof of Proposition 2.** We have proved in Section 1.3 that for every $f \in C_0^\infty(\mathbb{R}^d)$ there exists $U \in \mathcal{L}(C_0^\infty(\mathbb{R}^d))$ such that $U \phi \in C_0^\infty(\mathbb{R}^d)$ (see (8)) and $C_0^\infty(\mathbb{R}^d)$ in dense in $C_0^\infty(\mathbb{R}^d)$, we can extend $U$ to $C_0^\infty(\mathbb{R}^d)$ by continuity. Thus, property (E2) holds for all $(s, t) \in D_T$.

Property (E1) for $(s, t) \in D_T$ follows from the uniform convergence and the composition property of $(U_m(t, s))_{0 \leq s \leq t}$.

Property (E3) for $(s, t) \in D_T$ follows from estimate (6) and the fact that $U_m(t, s)$ preserve positivity.

Since $T > 0$ was chosen arbitrarily, operators $U(t, s) \in \mathcal{L}(C_0^\infty(\mathbb{R}^d))$ are well defined and satisfy (E1)-(E3) for all $0 \leq s \leq t < \infty$. 

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We are left to prove (E4). Fix $f \in C_\infty(\mathbb{R}^d)$, and set $u(t, s, \cdot) = U(t, s)f(\cdot)$. Since $b \in L^2_{\text{loc}}((s, \infty) \times \mathbb{R}^d, \mathbb{R}^d)$, it suffices to show that $\nabla u \in L^2_{\text{loc}}((s, \infty) \times \mathbb{R}^d, \mathbb{R}^d)$ in order to conclude $b \cdot \nabla u \in L^1_{\text{loc}}((s, \infty) \times \mathbb{R}^d)$. We argue as follows. Define, for a fixed $k > \frac{d}{2}$,

$$\rho_\delta(x) := (1 + \delta |x|^2)^{-k}, \quad \delta > 0, \quad x \in \mathbb{R}^d.$$ 

Then $\rho_\delta \in L^1(\mathbb{R}^d)$, and

$$|\nabla \rho_\delta(x)| \leq 2k\sqrt{\delta}\rho_\delta(x), \quad \left| \frac{\nabla \rho_\delta(x)}{\rho_\delta(x)} \right| \leq 4k^2\delta\rho_\delta(x) \quad \text{for all } x \in \mathbb{R}^d, \quad 0 < \delta < 1. \quad (45)$$

Let $u_m(t, s, \cdot) = U_m(t, s)f(\cdot)$.

**Claim 9.** For any $\delta > 0$ sufficiently small there exist constants $c_1, c_2 < \infty$ such that for all $m$

$$\int_s^T \langle (\nabla u_m)^2 \rho_\delta \rangle dt \leq c_1 \langle f^2 \rho_\delta \rangle + c_2 \|f\|_\infty^2, \quad (s, \tau) \in D_T, \quad f \in C_\infty(\mathbb{R}^d).$$ 

*Proof of Claim 9.* There exists $h$ in (10) such that for all $m$

$$C_0 \int_s^T \langle (\nabla v_m)^2 \rho_\delta \rangle dt \leq \langle f^2 \rho_\delta \rangle + C_1 k\sqrt{\delta} \int_s^T \langle v_m^2 \rho_\delta \rangle dt + \int_s^T \langle (\nabla v_m)^2 \rho_\delta \rangle dt, \quad (s, \tau) \in D_T. \quad (47)$$

where $0 < C_0 < \infty$, $C_1 < \infty$ are independent of $m$ and $\delta$. The proof goes along the lines of the proof of Claim 8(b) but with $1 - \rho_\delta$ replaced by $\rho_\delta$, and estimates (45) used in place of estimates (33). We obtain from (47)

$$(C_0 - C_1 k\sqrt{\delta}) \int_s^T \langle (\nabla v_m)^2 \rho_\delta \rangle dt \leq \langle f^2 \rho_\delta \rangle + C_1 k\sqrt{\delta} \int_s^T \langle v_m^2 \rho_\delta \rangle dt, \quad (s, \tau) \in D_T, \quad \text{for all } m, \quad (48)$$

where $C_0 - C_1 k\sqrt{\delta} > 0$ for $\delta > 0$ sufficiently small. By (12)

$$\sup_m \|v_m\|_{L^\infty(D_T \times \mathbb{R}^d)} \leq c \|f\|_\infty$$

for some $c < \infty$. Therefore, since $\rho_\delta \in L^1(\mathbb{R}^d)$, we have $\int_s^T \langle v_m^2 \rho_\delta \rangle dt \leq C_3 \|f\|_\infty^2, \quad C_3 := (\tau - s)c^2 \langle \rho_\delta \rangle < \infty$. Substituting this estimate into (48), which yields for all $m$

$$\int_s^T \langle (\nabla v_m)^2 \rho_\delta \rangle dt \leq (C_0 - C_1 k\sqrt{\delta})^{-1} \langle f^2 \rho_\delta \rangle + (C_0 - C_1 k\sqrt{\delta})^{-1} C_1 C_3 \|f\|_\infty^2, \quad (s, \tau) \in D_T,$$

and using identity (11), we obtain (16).

By Claim 9 sequence $\{\nabla u_m|_{[s, T] \times \bar{U}}\}$ is weakly relatively compact in $L^2([s, T] \times \bar{U}, \mathbb{R}^d)$, for any open bounded $U \subset \mathbb{R}^d$. Hence, $\nabla u|_{(s, T) \times \bar{U}}$ (understood in the sense of distributions) is in $L^2([s, T] \times \bar{U}, \mathbb{R}^d)$. Therefore, $\nabla u \in L^2_{\text{loc}}((s, \infty) \times \mathbb{R}^d, \mathbb{R}^d)$.

It remains to show that $u$ satisfies integral identity (3). Obviously, $u_m(t, s, \cdot) = U_m(t, s)f(\cdot)$ satisfy

$$\int_0^\infty \langle u_m, \dot{\psi} \rangle dt - \int_0^\infty \langle u_m, \Delta \psi \rangle dt - \int_0^\infty \langle b_m \cdot \nabla u_m, \psi \rangle dt = 0, \quad \psi \in C_0^\infty((s, \infty) \times \mathbb{R}^d).$$

The integral identity (3) will follow upon taking $m \to \infty$ (possibly for a subsequence of $\{u_m\}$); to justify taking the limit, we argue as follows. Fix some $\psi$, without loss of generality $\text{supp} \psi \subset (s, T) \times U$ for an open bounded $U \subset \mathbb{R}^d$. We can re-write the last identity as

$$\int_s^T \langle u_m, \dot{\psi} \rangle dt - \int_s^T \langle u_m, \Delta \psi \rangle dt - \int_s^T \langle (b_m - b) \cdot \nabla u_m, \psi \rangle dt - \int_s^T \langle b \cdot \nabla u_m, \psi \rangle dt = 0. \quad (49)$$
Since \( u_m \to u \) in \( C(D_T, C_\infty (\mathbb{R}^d)) \), and \( \psi \) has compact support, we can pass to the limit in the first two terms in the left-hand side of (49). By Hölder inequality

\[
\left| \int_s^T (b_m - b) \cdot \nabla u_m, \psi \right| dt \leq C_0^2 \left( \int_s^T (b_m - b)^2 |\psi| \right)^{\frac{1}{2}} , \quad C_0 := \sup \int_s^T (|\nabla u_m|^2 |\psi|) dt,
\]

where \( C_0 < \infty \) by Claim 9. Since \( b_m \to b \) in \( L^2_{loc}([0, \infty) \times \mathbb{R}^d) \) by (C1), the third term tends to 0 as \( m \to \infty \). Finally, we can pass to the limit in the fourth term in (49) because \( \{\nabla u_m\}_{[s,T] \times U} \) is weakly relatively compact in \( L^2([s,T] \times U) \), see Claim 9 and \( b \psi \in L^2([0, T] \times U) \).

4. Appendix

Proof of (13). Let us show that there is \( h \) in (10) such that \( \|v_m(\tau, s, \cdot)\|_p \leq \|f\|_p, (s, \tau) \in D \), for all \( m \). Then by (11) \( \|u_m(\tau, s, \cdot)\|_p \leq C\|f\|_p \), and (13) follows by the weak relative compactness in \( L^p(\mathbb{R}^d) \).

Without loss of generality \( f \geq 0 \), so \( v_m \geq 0 \). Multiply (10) by \( v_m^p - 1 \) and integrate to get

\[
R := \int_s^T \langle v_{m}^p, \partial_t v_m \rangle dt = \int_s^T \langle v_{m}^{p-1}, \Delta v_m \rangle dt + \int_s^T \langle v_{m}^{p-1}, b_m \cdot \nabla v \rangle dt + \int_s^T h'(t) \langle v_{m}^p \rangle dt
= : R_1 + R_2 + R_3.
\]

We have

\[
R = \frac{1}{p} \langle v_{m}^p, \tau, s, \cdot \rangle - \frac{1}{p} \langle f^p \rangle, \quad R_1 = -(p - 1) \frac{4}{p^2} \int_s^T (|\nabla v_m|^2)^2 dt,
\]

and, using elementary inequality \( ab \leq \eta a^2 + \frac{1}{4\eta} b^2 (\eta > 0) \) and condition (C2),

\[
R_2 = \frac{2}{p} \int_s^T \langle v_{m}^{p-1}, b_m \cdot \nabla v_m \rangle dt \leq \frac{2}{p} \eta \int_s^T (|\nabla v_m|^2)^2 dt + \frac{1}{2p^2} \beta \int_s^T (|\nabla v_m|^2)^2 dt + \int_s^T \langle g(t) v_m^p \rangle dt.
\]

Therefore,

\[
\frac{1}{p} \langle v_{m}^p, \tau, s, \cdot \rangle + \left( \frac{4(p - 1)}{p^2} - \frac{2}{p} \eta + \frac{\beta}{2p^2} \right) \int_s^T (|\nabla v_m|^2)^2 dt
\leq \frac{1}{p} \langle f^p \rangle + \int_s^T \left( h'(t) + \frac{\beta}{2p^2} g(t) \right) \langle v_{m}^p \rangle dt
\]

The maximum of \( \eta \mapsto \frac{4(p - 1)}{p^2} - \frac{2}{p} \eta - \frac{\beta}{2p^2} \), attained at \( \eta_* = \sqrt{\beta/4} \), is positive if and only if \( p > (1 - \sqrt{\beta/4})^{-1} \). Set \( \eta = \eta_* \), \( h(t) := \frac{\beta}{2p^2} \int_0^t g(\theta) d\theta \) to obtain \( \langle v_{m}^p, \tau, s, \cdot \rangle \leq \langle f^p \rangle \), i.e. (13). \( \square \)

References


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