[5pt] 1. Use Laplace transform to solve the following initial value problem

$$x'' - 4x = 0,\ x(0) = 0,\ x'(0) = 1.$$

Solution.

Denote $\mathcal{L}(x(t)) = X(s)$. Then

$$\mathcal{L}(x'(t)) = sX(s),\ \mathcal{L}(x''(t)) = s^2X(s) - 1.$$

Taking Laplace transform of the above equation, we get $(s^2 - 4)X(s) = 1$, or alternatively $X(s) = \dfrac{1}{(s-2)(s+2)}$. By means of partial fraction expansion,

$$\frac{1}{(s-2)(s+2)} = \frac{A}{s-2} + \frac{B}{s+2}.$$

Multiplying both sides by the denominator and equating coefficients, we obtain $1 = A(s+2) + B(s-2)$, substitute $s = 2 \Rightarrow A = \frac{1}{4}$, $s = -2 \Rightarrow B = -\frac{1}{4}$. Hence $X(s) = \frac{1}{4}\frac{1}{s-2} - \frac{1}{4}\frac{1}{s+2}$, which corresponds to

$$x(t) = \frac{1}{4}(\mathbf{e}^{2t} - \mathbf{e}^{-2t})$$

or alternatively

$$x(t) = \frac{1}{2}\sinh 2t.$$

Answer. $\frac{1}{2}\sinh 2t$.

[5pt] 2. Use Laplace transform to solve the following initial value problem

$$x'' - 2x' + x = e^t,\ x(0) = 0,\ x'(0) = 0.$$

Solution.

Denoting $\mathcal{L}(x(t)) = X(s)$ we get

$$\mathcal{L}(x'(t)) = sX(s),\ \mathcal{L}(x''(t)) = s^2X(s).$$

Hence $s^2X - 2sX + X = \dfrac{1}{s-1}$, or $X(s) = \dfrac{1}{(s-1)^3}$. Since $t^2 \sim \dfrac{2!}{s^3} = \dfrac{2}{s^3}$, we have $\dfrac{1}{s^3} \sim \frac{t^2}{2}$ and by translation theorem $\dfrac{1}{(s-1)^3} \sim \frac{1}{2}t^2\mathbf{e}^t$.

Answer. $t^2\mathbf{e}^t/2$.