GIVEN ①:-

$\forall \varepsilon_1 > 0, \exists \delta_1 > 0$ s.t. $0 < |x - a| < \delta_1 \Rightarrow |g(x) - b| < \varepsilon_1$

GIVEN ②:-

$\forall \varepsilon_2 > 0, \exists \delta_2 > 0$ s.t. $0 < |y - b| < \delta_2 \Rightarrow |f(y) - f(b)| < \varepsilon_2$

WTS:-

$\forall \varepsilon_3 > 0, \exists \delta_3 > 0$ s.t. $0 < |x - a| < \delta_3 \Rightarrow |f(g(x)) - f(a)| < \varepsilon_3$

Proof:-

Let $\varepsilon_3 > 0$

I choose $\varepsilon_1 = \delta_2$ in def. of $\lim_{x \to a} g(x) = b$:-

$$0 < |x - a| < \delta_1 \Rightarrow |g(x) - b| < \delta_2 \quad \text{-(i)}$$

Now, I chose $\varepsilon_2 = \varepsilon_3$ in my definition of 'f being cts @ b':-

$$0 < |y - b| < \delta_2 \Rightarrow |f(y) - f(b)| < \varepsilon_3$$

Let $y = g(x)$

$$\therefore 0 < |g(x) - b| < \delta_2 \Rightarrow |f(g(x)) - f(b)| < \varepsilon_3 \quad \text{(ii)}$$

~~From (i), we can conclude that there exists a~~

From (i) and (ii),

$$0 < |x - a| < \delta_1 \Rightarrow |f(g(x)) - f(b)| < \varepsilon_3 \quad \text{-(iii)}$$

$\therefore$ for all values of ε_3 there exists a $\delta_3 (= \delta_1)$ s.t. (iii) is true

$\therefore$ from defn of limit, $\lim_{x \to a} f(g(x)) = f(a)$ ∎

Comment 1: Proof writing error

The second line says "I choose $\epsilon_1 = \delta_2$". At this point in the proof, only ϵ_3 has been defined (in the first line, as a given ϵ_3). I, as the reader, have no idea what δ_2 is and what it has to do with that given ϵ_3. It is only afterwards that I figure out you mean a δ_2 which works for given (2) when we set $\epsilon_2 = \epsilon_3$. In a proof, before you start using a variable (in this case setting things to be equal to the variable δ_2), you need to make sure the reader knows what it is sfirst.

What should actually be done is choosing $\epsilon_2 = \epsilon_3$ first, which you can do since your first line already talked about ϵ_3. Then using given (2) to choose a δ_2 which works for that choice of ϵ_2. Then setting $\epsilon_1 = \delta_2$, since now we know how δ_2 is defined.

Comment 2: Technical error

(i) says $0 < |x - a| < \delta_1 \implies |g(x) - b| < \delta_2$. (ii) says $0 < |g(x) - b| < \delta_2 \implies |f(g(x)) - f(b)| < \epsilon_3$. The goal was to use (i) and (ii) together to produce the result. However, $|g(x) - b| < \delta_2$ does not imply $0 < |g(x) - b| < \delta_2$. For example, what if $g(x) = b$. Fortunately, the condition given to us is that f is continuous at b, which means we can change given (2) and (ii) to a slightly different statement for which this won't be an issue. What is that statement?

Comment 3: Proof writing error

In both (i) and (ii) in the proof, the δ_1 and δ_2 are unquantified. After you say, "I choose... in the def. of ....", you need to say, "then there exists a $\delta_1 > 0$ (or $\delta_2 > 0$) s.t. ... ".