Announcements

- **Topics:** Exponentials and logarithms, inverse trig, extrema, Rolle's Theorem
- **Homework:** Watch videos 5.7 - 5.12, 6.1 and 6.2.

Warm-up: Logarithm and Absolute Value

The function F is defined by the equation

$$F(x) = \ln |x|.$$

What is its derivative?

1. $F'(x) = \dfrac{1}{x}$

2. $F'(x) = \dfrac{1}{|x|}$

3. F is not differentiable on its domain

Warm up

Compute the derivative of the following functions:

1. $f(x) = e^{\sin x + \cos x} \ln x$
2. $f(x) = \pi^{\tan x}$
3. $f(x) = \ln\left[e^x + \ln \ln \ln x\right]$

Reminder: We know:

- $\dfrac{d}{dx} e^x = e^x$
- $\dfrac{d}{dx} a^x = a^x \ln a$
- $\dfrac{d}{dx} \ln x = \dfrac{1}{x}$

Multiple choice

The derivative of x^x is:

a. $x(x^{x-1})$

b. $(ln(x)+1)x^x$

b. $ln(x)x^x$

Logrithmic differentiation

Find $\frac{dy}{dx}$:

1. $y = x^{x^x} + 1$
2. $x^y = x^2 + y^x$

Exercise: Hard derivatives made easier

Calculate the derivative of

$$h(x) = \sqrt[3]{\frac{\left(\sin^6 x\right)\sqrt{x^7 + 6x + 2}}{3^x\left(x^{10} + 2x\right)^{10}}}$$

Hint: Differentiate $\ln(h(x))$ instead.

A different type of logarithm

Calculate the derivative of

$$f(x) = \log_{x+1}(x^2 + 1)$$

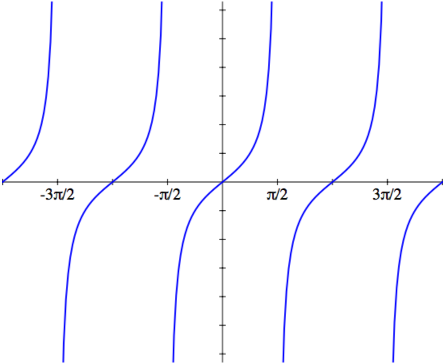
Note: This is a new function. We have not given you a formula for it yet, That is on purpose.

Hint: If you do not know where to start, remember the definition of logarithm:

$$\log_a b = c \iff a^c = b.$$

The arctan function

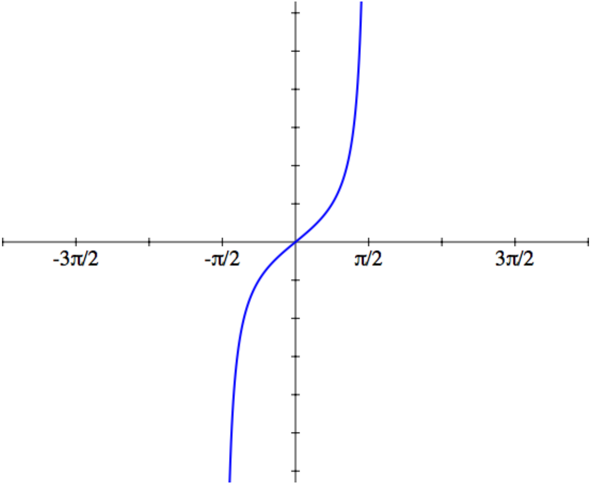
Here's (part of) the graph of the tan function.



Question. Does this function have an inverse?
Problem. Find the largest interval containing 0 such that the restriction of tan to it is injective.

The arctan function

We define arctan to be the inverse of the function with this graph:



The arctan function

In symbols, that means we define the function arctan as the inverse of the function

$$g(x) = \tan x, \text{ restricted to the interval } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

In other words, if $x, y \in \mathbb{R}$, then

$$\arctan(y) = x \quad \iff \quad \begin{cases} ??? \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \\ ??? \end{cases}$$

Problem 1. What should be where the question marks are?

Problem 2. What are the domain and range of arctan?

Problem 3. Sketch the graph of arctan.

The arctan function

To remind you:

$$\arctan(y) = x \quad \iff \quad \begin{cases} x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \\ \tan x = y \end{cases}$$

Compute the following values:

1. $\arctan\left(\tan\left(1\right)\right)$
2. $\arctan\left(\tan\left(3\right)\right)$
3. $\arctan\left(\tan\left(\frac{\pi}{2}\right)\right)$
4. $\arctan\left(\tan\left(-6\right)\right)$
5. $\tan\left(\arctan\left(0\right)\right)$
6. $\tan\left(\arctan\left(10\right)\right)$

Derivative of arctan

Compute

$$\frac{d}{dx}\arctan(x).$$

Standard choice of restrictions

We make the following standard choice of restrictions when we define the inverse trig functions:

1. $\sin(x)$ restricted to $[-\frac{\pi}{2}, \frac{\pi}{2}]$.
2. $\cos(x)$ restricted to $[0, \pi]$.
3. $\tan(x)$ restricted to $(-\frac{\pi}{2}, \frac{\pi}{2})$.
4. $\sec(x)$ restricted to $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$.
5. $\csc(x)$ restricted to $[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$.
6. $\cot(x)$ restricted to $(0, \pi)$.

Developing $\arctan_2$

Let's define $\arctan_2(x)$ as the inverse of the restriction of $\tan(x)$ to the interval $(\frac{\pi}{2}, \frac{3\pi}{2})$. Find the following:

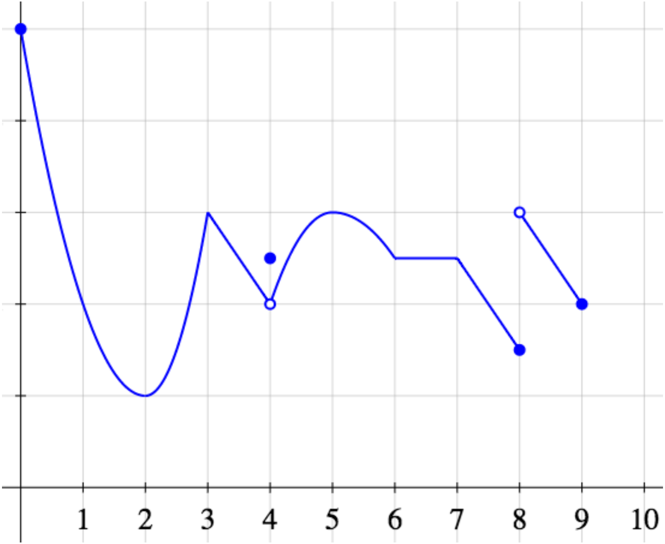
1. The domain and the range of $\arctan_2$.

2. A graph of $\arctan_2$.

3. $\tan(\arctan_2(12))$, $\arctan_2(\tan(0))$, $\arctan_2(\tan(\pi))$, $\arctan_2(\tan(7))$

4. Compute the derivative of $\arctan_2$.

Find local and global extrema of the function with this graph:



What can you conclude?

We know the following about the function f.

- f has domain $\mathbb{R}$.
- f is continuous
- $f(0) = 0$
- For every $x \in \mathbb{R}$, $f(x) \geq x$.

What can you conclude about $f'(0)$? Prove it.

Hint: Sketch the graph of f. Looking at the graph, make a conjecture.
To prove it, imitate the proof of the Local EVT from Video 5.3.

Fractional exponents

Let $g(x) = x^{2/3}(x-1)^3$.

Find local and global extrema of g on $[-1, 2]$.

Trig extrema

Let $f(x) = \dfrac{\sin x}{3 + \cos x}$.

Find the maximum and minimum values of f.

Zeroes of the derivative

For each of the following conditions, sketch the graph of some function f that is differentiable on $\mathbb{R}$ and such that

1. f has exactly 3 zeroes and f' has exactly 2 zeroes.
2. f has exactly 3 zeroes and f' has exactly 3 zeroes.
3. f has exactly 3 zeroes and f' has exactly 1 zero.
4. f has exactly 3 zeroes and f' has infinitely many zeroes.

How many zeroes?

Let

$$f(x) = e^x - \sin x + x^2 + 10x$$

How many zeroes does f have? **Hint:** Differentiate. Is it obvious how many zeroes the derivative has? If not, differentiate again.

Zeroes of a polynomial

You probably learned in high school that a polynomial of degree n has at most n real zeroes. Now you can prove it!
Hint: Use induction.