Announcements

- **Topics:** Implicit differentiation, inverses, exponentials and logarithms
- **Homework:** Watch videos 4.12 - 4.14, 5.1 - 5.6

A pesky function

Let $h(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.

1. Calculate $h'(x)$ for any $x \neq 0$.
2. Using the definition of derivative, calculate $h'(0)$.
3. Calculate $\lim_{x \to 0} h'(x)$

   Hint: Questions 2 and 3 have different answers.
4. Is h continuous at 0?
5. Is h differentiable at 0?
6. Is h' continuous at 0?

Differentiable functions

Let $a \in \mathbb{R}$.
Let f be a function with domain $\mathbb{R}$.
Assume f is differentiable everywhere.
What can we conclude?

1. $f(a)$ is defined.
2. $\lim_{x \to a} f(x)$ exists.
3. f is continuous at a.
4. $f'(a)$ exists.
5. $\lim_{x \to a} f'(x)$ exists.
6. f' is continuous at a.

Implicit differentiation

The equation

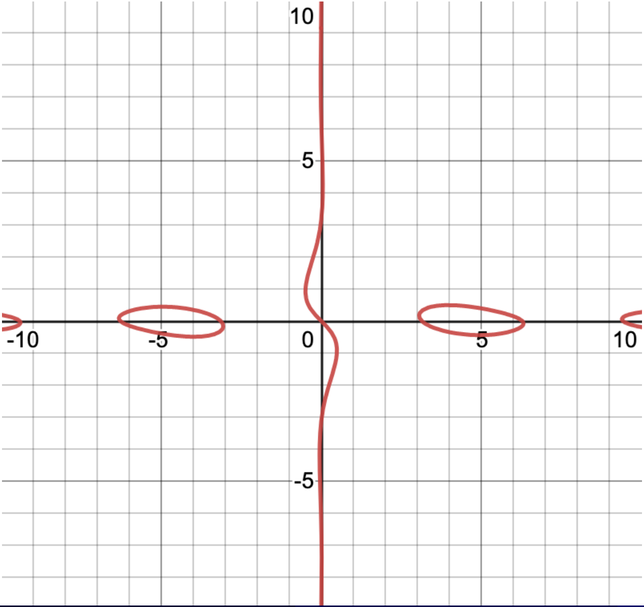
$$\sin(x+y) + xy^2 = 0$$

defines a function $y = h(x)$ near $(0, 0)$. ▸ graph

Using implicit differentiation, compute

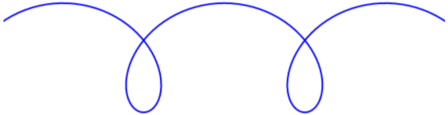
1. $h(0)$
2. $h'(0)$
3. $h''(0)$
4. $h'''(0)$

Graph



Worm up

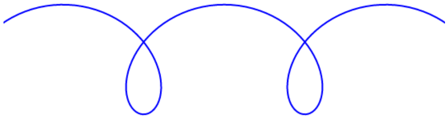
A worm is crawling accross the table. The path of the worm looks something like this:



True or False?

The position of the worm is a function.

Worm function



A worm is crawling accross the table.
For any time t, let $f(t)$ be the position of the worm.
This defines a function f.

1. What is the domain of f?
2. What is the codomain of f?
3. What is the range of f?

Composition and inverses

Assume for simplicity that all functions in this problem have domain $\mathbb{R}$.

Let f and g be functions. Assume they each have an inverse.

Is $(f \circ g)^{-1} = f^{-1} \circ g^{-1}$?

- If YES, prove it.
- If NO, fix the statement.

If you do not know how to start, experiment with the functions

$$f(x) = x + 1, \qquad g(x) = 2x.$$

Composition of one-to-one functions

Assume for simplicity that all functions in this problem have domain $\mathbb{R}$. Prove the following theorem.

Theorem A

Let f and g be functions.
IF f and g are one-to-one,
THEN $f \circ g$ is one-to-one.

Suggestion:

1. Write the definition of what you want to prove.
2. Figure out the formal structure of the proof.
3. Complete the proof (use the hypotheses!)

Composition of one-to-one functions – 2

Assume for simplicity that all functions in this problem have domain $\mathbb{R}$. Prove the following theorem.

Theorem B

Let f and g be functions.
IF $f \circ g$ is one-to-one, THEN g is one-to-one.

Suggestion:

1. Transform the "$P \implies Q$" theorem into an equivalent "(not Q) $\implies$ (not P)" theorem.
   You will prove that one instead.
2. Write the definition of the hypotheses and of the conclusion.
3. Write the proof.

Composition of one-to-one functions – 3

Assume for simplicity that all functions in this problem have domain $\mathbb{R}$.

Prove the following claim is FALSE.

Claim

Let f and g be functions.
IF $f \circ g$ is one-to-one,
THEN f is one-to-one.

Absolute value and inverses

Let

$$h(x) = x|x| + 1$$

1. Calculate $h^{-1}(-8)$.
2. Sketch the graph of h.
3. Find an equation for h^{-1}.
4. Sketch the graph of h^{-1}.
5. Verify that
   - for every $t \in \boxed{???}$, $\quad h(h^{-1}(t)) = t$.
   - for every $t \in \boxed{???}$, $\quad h^{-1}(h(t)) = t$.

Derivatives of the inverse function

Let f be a one-to-one function.
Let $a, b \in \mathbb{R}$ such that $b = f(a)$.

1. Obtain a formula for $\left(f^{-1}\right)'(b)$ in terms of $f'(a)$.
   Hint: This was done in Video 4.4
2. Obtain a formula for $\left(f^{-1}\right)''(b)$ in terms of $f'(a)$ and $f''(a)$.
3. *Challenge:* Obtain a formula for $\left(f^{-1}\right)'''(b)$ in terms of $f'(a)$, $f''(a)$, and $f'''(a)$.