Announcements

- **Topics:** Continuity, behaviour of limits under composition, computations
- **Homework:** Watch videos 2.21, 2.22, 3.1 - 3.5, 3.8

Continuity

For a function f defined on an open interval of a, we say $f(x)$ is cts at a iff

Definition 1

$\lim_{x \to a} f(x) = f(a)$

This is clearly equivalently to

Definition 2

$\forall \epsilon > 0, \exists \delta > 0$ s.t. $0 < |x - a| < \delta \implies |f(x) - f(a)| < \epsilon.$

Slightly less clearly, it is also equivalent to

Definition 3

$\forall \epsilon > 0, \exists \delta > 0$ s.t. $|x - a| < \delta \implies |f(x) - f(a)| < \epsilon.$

Continuity on different sets

Continuous at a point

f continuous at c means $\lim_{x \to c} f(x) = f(c)$.

Continuous on an open interval

f continuous on the interval (a, b) means $\forall c \in (a, b)$, f is continuous at c.

Continuous on a closed interval

f continuous on the interval $[a, b]$ means

1. $\lim_{x \to a^+} f(x) = f(a)$
2. $\forall c \in (a, b)$, f is continuous at c
3. $\lim_{x \to b^-} f(x) = f(b)$

Undefined function

Let $a \in \mathbb{R}$ and let f be a function. Assume $f(a)$ is undefined.

What can we conclude?

1. $\lim_{x \to a} f(x)$ exist
2. $\lim_{x \to a} f(x)$ doesn't exist.
3. No conclusion. $\lim_{x \to a} f(x)$ may or may not exist.

What else can we conclude?

4. f is continuous at a.
5. f is not continuous at a.
6. No conclusion. f may or may not be continuous at a.

A new function

- Let $x, y \in \mathbb{R}$. What does the following expression calculate? Prove it.

$$f(x, y) = \frac{x + y + |x - y|}{2}$$

  Suggestion: If you don't know how to start, try some sample values of x and y.

- Write a similar expression to compute $\min\{x, y\}$.

More continuous functions

We want to prove the following theorem

Theorem

IF f and g are continuous functions
THEN $h(x) = \max\{f(x), g(x)\}$ is also a continuous function.

You are allowed to use all results that we already know.
What is the fastest way to prove this?
Hint: There is a way to prove this quickly without writing any epsilons.

True or False? – Discontinuities

1. IF f and g have removable discontinuities at a
THEN $f + g$ has a removable discontinuity at a

2. IF f and g have non-removable discontinuities at a
THEN $f + g$ has a non-removable discontinuity at a

Dirichlet function

Consider the Dirichlet function

$$D(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \backslash \mathbb{Q} \end{cases}$$

1. Write the definition of $\lim_{x \to 0} D(x) \neq 0.5$.
2. Prove it.
3. Write the definition of $\lim_{x \to 0} D(x)$ DNE.
4. Exercise: Prove 3.

Continuity examples

Find examples of a function defined on $\mathbb{R}$ satisfying the following conditions:

1. $f(x)$ is continuous on $\mathbb{R}$.

2. $g(x)$ is continuous at every $c \in \mathbb{R} \setminus \{0\}$ and discontinuous at 0.

3. $h(x)$ is discontinuous at every $c \in \mathbb{R}$.

4. $m(x)$ is continuous at 0 and discontinuous at every $c \in \mathbb{R}$.

Hint: Try adjusting the Dirichlet function.

Which one is the correct claim?

Claim 1?

(Assuming these limits exist)

$$\lim_{x \to a} g(f(x)) \; = \; g\left(\lim_{x \to a} f(x)\right)$$

Claim 2?

IF (A) $\lim_{x \to a} f(x) = L,$ and (B) $\lim_{t \to L} g(t) = M$

THEN (C) $\lim_{x \to a} g(f(x)) = M$

A difficult example

Construct a pair of functions f and g such that

$$\begin{aligned}\lim_{x\to 0} f(x) &= 1\\ \lim_{t\to 1} g(t) &= 2\\ \lim_{x\to 0} g(f(x)) &= 42\end{aligned}$$

A theorem about limits

Let f be a function with domain $\mathbb{R}$ such that

$$\lim_{x \to 0} f(x) = 3$$

Prove that

$$\lim_{x \to 0} [5f(2x)] = 15$$

directly from the definition of limit. Do not use any of the limit laws.

1. Write down the formal definition of the statement you want to prove.
2. Write down what the structure of the formal proof should be, without filling the details.
3. Rough work.
4. Write down a complete proof.

Computation warmup

Compute the following

1. $\lim_{x\to\infty} x\sin(\frac{1}{x})$
2. $\lim_{x\to\infty} \frac{1}{x}\sin(x)$

Transforming limits

The only thing we know about the function g is that

$$\lim_{x\to 0} \frac{g(x)}{x^2} = 2.$$

Use it to compute the following limits:

1. $\displaystyle\lim_{x\to 0} \frac{g(x)}{x}$
2. $\displaystyle\lim_{x\to 0} \frac{g(x)}{x^4}$
3. $\displaystyle\lim_{x\to 0} \frac{g(3x)}{x^2}$

Trig computations

Using that $\lim_{x\to 0} \frac{\sin x}{x} = 1$, compute the following limits:

1. $\lim_{x\to 2} \frac{\sin x}{x}$
2. $\lim_{x\to 0} \frac{\sin(5x)}{x}$
3. $\lim_{x\to 0} \frac{\tan^2(2x^2)}{x^4}$
4. $\lim_{x\to 0} \frac{\sin e^x}{e^x}$
5. $\lim_{x\to 0} \frac{1-\cos x}{x}$
6. $\lim_{x\to 0} \frac{\tan^{10}(2x^{20})}{\sin^{200}(3x)}$
7. $\lim_{x\to 0} \left[(\sin x)\,(\cos(2x))\,(\tan(3x))\,(\sec(4x))\,(\csc(5x))\,(\cot(6x))\right]$

Limits at infinity

Compute:

1. $\lim_{x\to\infty} \left(x^7 - 2x^5 + 11\right)$
2. $\lim_{x\to\infty} \left(x^2 - \sqrt{x^5+1}\right)$
3. $\lim_{x\to\infty} \frac{x^2+11}{x+1}$
4. $\lim_{x\to\infty} \frac{x^2+2x+3}{3x^2+4x+5}$
5. $\lim_{x\to\infty} \frac{x^3+\sqrt{2x^6+1}}{2x^3+\sqrt{x^5+1}}$

Plus or minus infinity?

Compute:

1. $$\lim_{x \to -3^+} \frac{x^2 - 9}{3 - 2x - x^2}$$
2. $$\lim_{x \to 1^+} \frac{x^2 - 9}{3 - 2x - x^2}$$

Is this computation correct?

Compute $L = \lim_{x \to -\infty} \left[x - \sqrt{x^2 + x} \right]$.

Solution 1

$$L = \lim_{x \to -\infty} \frac{\left[x - \sqrt{x^2 + x} \right] \left[x + \sqrt{x^2 + x} \right]}{\left[x + \sqrt{x^2 + x} \right]} = \lim_{x \to -\infty} \frac{x^2 - (x^2 + x)}{\left[x + \sqrt{x^2 + x} \right]}$$

$$= \lim_{x \to -\infty} \frac{-x}{x \left[1 + \sqrt{1 + \frac{1}{x}} \right]} = \lim_{x \to -\infty} \frac{-1}{\left[1 + \sqrt{1 + \frac{1}{x}} \right]} = \frac{-1}{2}$$