Announcements

- **Topics:** Formal proofs with limit definition and limit laws
- **Homework:** Watch videos 2.14 - 2.20.

Infinite limits

Definition

Let $a \in \mathbb{R}$.
Let f be a function defined at least on an interval around a, except possibly at a.
Write a formal definition for

$$\lim_{x \to a} f(x) = \infty.$$

Infinite limits

Which one(s) is the definition of $\lim_{x \to a} f(x) = \infty$?

1. $\forall M \in \mathbb{R},\ \exists \delta > 0 \text{ s.t. } 0 < |x - a| < \delta \implies f(x) > M$
2. $\forall M \in \mathbb{Z},\ \exists \delta > 0 \text{ s.t. } 0 < |x - a| < \delta \implies f(x) > M$
3. $\forall M > 0,\ \exists \delta > 0 \text{ s.t. } 0 < |x - a| < \delta \implies f(x) > M$
4. $\forall M > 5,\ \exists \delta > 0 \text{ s.t. } 0 < |x - a| < \delta \implies f(x) > M$

Related implications

Let $a \in \mathbb{R}$. Let f be a function. Assume we know

$$0 < |x - a| < 0.1 \quad \Longrightarrow \quad f(x) > 100$$

1. Which values of $M \in \mathbb{R}$ satisfy ... ?

$$0 < |x - a| < 0.1 \quad \Longrightarrow \quad f(x) > M$$

2. Which values of $\delta > 0$ satisfy ... ?

$$0 < |x - a| < \delta \quad \Longrightarrow \quad f(x) > 100$$

Warm-up

Let $x \in \mathbb{R}$ and S_1, S_2, S_3 and S_4 be logic statements. Suppose you know:

1. If $|x - 2| < 4$, then S_1 (is true).
2. If $|x - 2| < 5$, then S_2 (is true).

What condition do you need to guarantee S_1 and S_2 are both true?

Suppose you know:

1. If $x > 100$, then S_3 (is true).
2. If $x > 1000$, then S_4 (is true).

What condition do you need to guarantee S_3 and S_4 are both true?

Warm-up

1. Find a value of $\delta > 0$ s.t.

$$|x-2| < \delta \implies |2x-4| < 1.$$

2. Find all values of $\delta > 0$ s.t.

$$|x-2| < \delta \implies |2x-4| < 1.$$

3. Find all values of $\delta > 0$ s.t.

$$|x-2| < \delta \implies |2x-4| < 0.1.$$

3. Let $\epsilon > 0$, find all values of $\delta > 0$ s.t.

$$|x-2| < \delta \implies |2x-4| < \epsilon.$$

An $\epsilon - \delta$ proof

Goal

Prove

$$\lim_{x \to 2} 2x = 4$$

from the definition.

1. Write down the formal definition of claim. This is the statement you will need to prove.
2. Write down the structure of the proof without details.
3. Write down the complete proof.

Is this proof correct?

Claim

$\lim_{x \to 2} x^2 = 4$

Proof:

Let $\epsilon > 0$.
Choose $\delta = \frac{\epsilon}{|x+2|}$.
Let $x \in \mathbb{R}$.
Asssume $0 < |x-2| < \delta$, then,

$$|x^2 - 4| = |x-2||x+2| < \frac{\epsilon}{|x+2|}|x+2| = \epsilon.$$

□

Steps for doing an $\epsilon - \delta$ proofs: rough work

Goal

Prove

$$\lim_{x \to 2} x^2 = 4$$

from the definition.

1. Write down the formal definition of claim. What can you control? What are you trying to control? What is out of your control?

2. Start with the $|f(x) - L|$ part of the defintion. Algebraically manipulate it to get several terms.

3. Determine which one of the terms you can make arbitrarily small by constraining $|x - a|$.

4. Bound all other terms by constants by constraining $|x - a|$.

An $\epsilon - \delta$ proof

Goal

Prove

$$\lim_{x \to 2} x^2 = 4$$

from the definition.

Another $\epsilon - \delta$ proof

Goal

Prove

$$\lim_{x \to 4} \frac{3}{x} = \frac{3}{4}$$

from the definition.

1. Write down the formal definition of claim.

2. Start with the $|f(x) - L|$ part of the defintion. Algebraically manipulate it to get several terms.

3. Determine which one of the terms you can make arbitrarily small by constraining $|x - a|$.

4. Bound all other terms by constants by constraining $|x - a|$.

5. Write down a formal proof.

True or False?

True or false?

Claim

Let $a \in \mathbb{R}$.

Let f and g be functions defined near a.

- IF $\lim_{x \to a} f(x)$ exists and $\lim_{x \to a} g(x)$ DNE,
- THEN $\lim_{x \to a} [f(x) + g(x)]$ *DNE*.

True or False?

True or false?

Claim

Let $a \in \mathbb{R}$.

Let f and g be functions defined near a.

- IF $\lim_{x \to a} f(x)$ DNE and $\lim_{x \to a} g(x)$ DNE,
- THEN $\lim_{x \to a} [f(x) + g(x)]$ *DNE*.

True or False?

True or false?

Claim

Let $a \in \mathbb{R}$.

Let f and g be functions defined near a.

- IF $\lim_{x \to a} f(x) = 0$,
- THEN $\lim_{x \to a} [f(x)g(x)] = 0$.

A new squeeze

This is the Squeeze Theorem, as you know it:

The (classical) Squeeze Theorem

Let $a, L \in \mathbb{R}$.
Let f, g, and h be functions defined near a, except possibly at a.

IF
- For x close to a but not a, $h(x) \leq g(x) \leq f(x)$
- $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} h(x) = L$

THEN
- $\lim_{x \to a} g(x) = L$

Come up with a new version of the theorem about limits being infinity. (The conclusion should be $\lim_{x \to a} g(x) = \infty$.)

Hint: Draw a picture for the classical Squeeze Theorem. Then draw a picture for the new theorem.

A new squeeze

A (new) Squeeze Theorem

Let $a \in \mathbb{R}$.
Let g and h be functions defined near a, except possibly at a.

IF
- For x close to a but not a, $h(x) \leq g(x)$
- $\lim_{x \to a} h(x) = \infty$

THEN
- $\lim_{x \to a} g(x) = \infty$

1. Replace the first hypothesis with a more precise mathematical statement.
2. Write down the definition of what you want to prove.
3. Write down the structure of the formal proof.
4. Rough work
5. Write down a complete, formal proof.