Announcements

- **Topics:** Negation, implications, simple proofs, inductive proofs
- **Homework:** Watch videos 2.1 - 2.6.

Negation, a harder example

Negation example

Negate "Every page in this book contains at least one word whose first and last letters both come alphabetically before M".

Breakout room - Indecisive function

Construct a function f that satisfies all of the following properties at once:

- The domain of f is $\mathbb{R}$.
- $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ such that

$$x < y \text{ and } f(x) < f(y)$$

- $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ such that

$$x < y \text{ and } f(x) > f(y)$$

Am I lying?

I tell you: "If you get 80% or more on your midterm, then I will give you a piece of chocolate."

In which of the following scenarios would I have lied (i.e. said something false)?

1. You get 80% on your test, and I give you a piece of chocolate.
2. You get 70% on your test, and I don't give you a piece of chocolate.
3. You get 100% on your test, and I don't give you a piece of chocolate.
4. You get 60% on your test, and I give you a piece of chocolate.
5. I give everybody a piece of chocolate.
6. You get 60% on your test.

Selection task

Every card on the table has a number on one side and a letter on the other side.

I tell you: "(For all the cards on the table.) If a card has a vowel on one side then it must have an even number on the other side."

You see 4 cards with "B", "7", "8", "A".

Which cards do you have to turn over to make sure I'm telling the truth?

The Wason selection task negation

What is the negation of the statement

"(For all the cards on the table.) If a card has a vowel on one side then it must have an even number on the other side"?

1. "If a card has a vowel on one side then it must have an odd number on the other side."
2. "If a card has a consonant on one side then it must have an even number on the other side."
3. "If a card has a consonant on one side then it must have an odd number on the other side."
4. "There is a card with a vowel on one side and an odd number on the other side."

Mathier conditionals

True or false?

1. $\forall x \in \mathbb{R},\ x > 0 \implies x \geq 0.$
2. $\forall x \in \mathbb{R},\ x \geq 0 \implies x > 0.$

Even and Odd

For $x \in \mathbb{R}$, give a mathematical definition for the statements "x is an even number". Do the same for the statement "x is an odd number".

Definition

Let $x \in \mathbb{R}$.

x is even $\iff$???

x is odd $\iff$???

A claim about odd and even numbers

Claim

The sum of any two odd numbers is even.

Bad proof

Claim

The sum of any two odd numbers is even.

Proof

1 is odd.

3 is odd.

$1 + 3 = 4$ is even.

Bad proof

Claim

The sum of any two odd numbers is even.

Proof

For all n:

$\text{EVEN} + \text{EVEN} = \text{EVEN}$

$\text{EVEN} + \text{ODD} = \text{ODD}$

$\text{ODD} + \text{ODD} = \text{EVEN}$

Bad proof

Claim

The sum of any two odd numbers is even.

Proof

$(2a+1)+(2b+1)=2a+2b+2$ is even.

Proof exercise

Exercise

Prove: The sum of any two odd numbers is even.

Blue and sad sets

Let $A \subseteq \mathbb{R}$. We define two new concepts:

- A is a *blue set* when $\exists M \in \mathbb{R}$ s.t. $\forall x \in A$, $x < M$.
- A is a *sad set* when $\exists M \in \mathbb{R}$ s..t $\forall x \in A$, $x \leq M$.

The following argument is WRONG. Explain Why.
"Being blue and being sad are not the same. For example, take the set $A = [0, 1]$ and $M = 1$. Then the set A is sad because for every $x \in A$, $x \leq M$. However, A is not blue because $x = 1 \in A$ but $x \not< M$."

Blue and sad sets

Let $A \subseteq \mathbb{R}$. We define two new concepts:

- A is a *blue set* when $\exists M \in \mathbb{R}$ s.t. $\forall x \in A$, $x < M$.
- A is a *sad set* when $\exists M \in \mathbb{R}$ s..t $\forall x \in A$, $x \leq M$.

Prove that a blue set is exactly the same thing as a sad set.

Hint: To prove that two definitions are equivalent. You have to two things. What are they?

Blue and sad sets

Let $A \subseteq \mathbb{R}$. We define two new concepts:

- A is a *blue set* when $\exists M \in \mathbb{R}$ s.t. $\forall x \in A$, $x < M$.
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Is the empty set blue?

One-to-one functions

Let f be a function with domain D.

f is *one-to-one* means that ...

- ... different inputs (x) ...
- ... must produce different outputs ($f(x)$).

Write a formal definition of “one-to-one”.

One-to-one functions

Definition: Let f be a function with domain D.

f is one-to-one means ...

1. $f(x_1) \neq f(x_2)$
2. $\exists x_1, x_2 \in D,\ f(x_1) \neq f(x_2)$
3. $\forall x_1, x_2 \in D,\ f(x_1) \neq f(x_2)$
4. $\forall x_1, x_2 \in D,\ x_1 \neq x_2,\ f(x_1) \neq f(x_2)$
5. $\forall x_1, x_2 \in D,\ x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$
6. $\forall x_1, x_2 \in D,\ f(x_1) \neq f(x_2) \implies x_1 \neq x_2$
7. $\forall x_1, x_2 \in D,\ f(x_1) = f(x_2) \implies x_1 = x_2$

One-to-one functions

Let f be a function with domain D.

What does each of the following mean?

1. $f(x_1) \neq f(x_2)$
2. $\exists x_1, x_2 \in D,\ f(x_1) \neq f(x_2)$
3. $\forall x_1, x_2 \in D,\ f(x_1) \neq f(x_2)$
4. $\forall x_1, x_2 \in D,\ x_1 \neq x_2,\ f(x_1) \neq f(x_2)$
5. $\forall x_1, x_2 \in D,\ x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$
6. $\forall x_1, x_2 \in D,\ f(x_1) \neq f(x_2) \implies x_1 \neq x_2$
7. $\forall x_1, x_2 \in D,\ f(x_1) = f(x_2) \implies x_1 = x_2$

Standard induction

Standard induction

We have a list of statements S_n dependent on natural numbers n. To prove $\forall n \in \mathbb{N}$, S_n (is true) using standard induction we need to prove the following two statements:

1. ???
2. ???

For example, S_n can be the the list of statements "$0 + 1 + 2 + ... + n = \frac{(n)(n+1)}{2}$" for every $n \in \mathbb{N}$.

What if?

If you managed to show the following instead, what can you prove?

Non-standard induction 1

1. S_3 is true.
2. "$\forall n > 0,\ S_n \implies S_{n+1}$" is true.

Non-standard induction 2

1. S_1 is true.
2. "$\forall n > 1,\ S_n \implies S_{n+1}$" is true.

If you managed to show the following instead, what can you prove?

Non-standard induction 3

1. S_1 is true.
2. "$\forall n > 0,\ S_n \implies S_{n+3}$" is true.

Nonstandard induction 4

1. S_7 is true.
2. "$\forall n > 3,\ S_{n+1} \implies S_n$" is true.

Variations on induction

We want to prove

$$\forall n \geq 1,\ S_n$$

So far we have proven

- S_1
- $\forall n \geq 1,\ S_n \implies S_{n+3}$.

What else do we need to do?

Variations on induction

We want to prove

$$\forall n \in \mathbb{Z}, \ S_n$$

So far we have proven

- S_1

What else do we need to do?

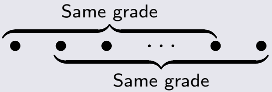
All 0s?

Theorem

$\forall N \in \mathbb{N}$, *in every set of N MAT137 students, those students will get the same grade.*

Proof.

- **Base case.** It is clearly true for $N = 1$.
- **Induction step.**
  Assume it is true for N. I'll show it is true for $N + 1$.
  Take a set of $N + 1$ students. By induction hypothesis:
  - The first N students have the same grade.
  - The last N students have the same grade.



Hence the $N + 1$ students all have the same grade as well.

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